Integrating both sides of this equation,

$$\log P_N(t) = -\mu t + A \qquad \text{...(23.38)}$$

To determine 'A', use the boundary condition $P_N(0) = 1$, and thus get $A = 0$ ($\because \log 1 = 0$).

Therefore, equation (23.38) becomes

$$\log P_N(t) = -\mu t \text{ or } P_N(t) = e^{-\mu t} \qquad \text{...(23.39)}$$

Step 2. In equation (23.36), put $n = N - 1$, and the value of $P_N(t)$ from equation (23.39),

$$P'_{N-1}(t) = -\mu P_{N-1}(t) + \mu P_N(t)$$

or $$P'_{N-1}(t) = -\mu P_{N-1}(t) + \mu e^{-\mu t} \qquad \text{[from equation (23.39)]}$$

or $$P'_{N-1}(t) + \mu P_{N-1}(t) = \mu e^{-\mu t} \qquad \text{...(23.40)}$$

The solution of this equation is given by

$$P_{N-1}(t)\, e^{\mu t} = \int \mu e^{-\mu t} e^{\mu t}\, dt + B \qquad (\because \text{I.F.} = e^{\mu t})$$

or $$P_{N-1}(t) = \mu t\, e^{-\mu t} + B e^{-\mu t} \qquad \text{...(23.41)}$$

To determine B, put $t = 0$, $P_{N-1}(t) = 0$ in (23.41) and get $B = 0$. Therefore,

$$P_{N-1}(t) = \frac{\mu t\, e^{-\mu t}}{1\,!}$$

Step 3. Putting $n = N - 2$ in equation (23.36) and proceeding exactly as in *Step 2*,

$$P_{N-2}(t) = \frac{e^{-\mu t}(\mu t)^2}{2\,!}$$

Step 4. Now, putting $n = N - 3, N - 4, \ldots, N - i$, and using induction process

$$P_{N-3}(t) = \frac{e^{-\mu t}(\mu t)^3}{3\,!}$$

$$\ldots \qquad \ldots$$

$$P_{N-i}(t) = \frac{e^{-\mu t}(\mu t)^i}{i\,!}, \quad i = 0, 1, 2, \ldots, N-1$$

$$\ldots \qquad \ldots$$

In general, on letting $n = N - i$ $$P_n(t) = \frac{e^{-\mu t}(\mu t)^{N-n}}{(N-n)\,!}, \quad n = 1, 2, \ldots, N \qquad \text{...(23.42)}$$

Step 5. In order to find $P_0(t)$, use the following procedure,

Since $$1 = \sum_{n=0}^{N} P_n(t) = P_0(t) + \sum_{n=1}^{N} P_n(t)$$

$\therefore$ $$P_0(t) = 1 - \sum_{n=1}^{N} P_n(t) = 1 - \sum_{n=1}^{N} \frac{(\mu t)^{N-n} e^{-\mu t}}{(N-n)\,!} \qquad \text{...(23.43)}$$

Finally, combining the results (23.42) and (23.43)

$$P_n(t) = \begin{cases} \dfrac{(\mu t)^{N-n} e^{-\mu t}}{(N-n)\,!}, & \text{for } n = 1, 2, \ldots, N \\[2ex] 1 - \displaystyle\sum_{n=1}^{N} \frac{(\mu t)^{N-n} e^{-\mu t}}{(N-n)\,!}, & \text{for } n = 0 \end{cases} \qquad \text{...(23.44)}$$

Thus, the number of departures in time t follows the '*Truncated Poisson Distribution*'.

Q. Establish the probability distribution formula for Pure-Death Process.

23.7-6. Derivation of Service Time Distribution

Let T be the random variable denoting the service time and t the possible value of T.

Let $S(t)$ and $s(t)$ be the *cumulative density function* and the *probability density function* of T, respectively.

To find $s(t)$ for the Poisson departure case, it has been observed that the probability of no service during time 0 to t is equivalent to the probability of having no departure during the same period.

Thus, Prob. [*service time* $T \geq t$] = Prob. [*no departure during* t] = $P_N(t)$

where there are N units in the system and no arrival is allowed after N. Therefore, $P_N(t) = e^{-\mu t}$

$$S(t) = \text{Prob. } (T \leq t) = 1 - \text{Prob. } [T \geq t] \quad \text{or} \quad S(t) = 1 - e^{-\mu t}$$

Differentiating both sides, w.r.t. 't', we get

$$\frac{d}{dt} S(t) = s(t) = \begin{cases} \mu e^{-\mu t}, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

Thus, it is concluded that the service time distribution is *'Exponential'* with mean $1/\mu$ and variance $1/\mu^2$.

Thus, mean service time = $1/\mu$.

Q. Explain the role of exponential distribution and its characteristics. **[Bhubneshwar (IT) 2004]**

23.7-7. Analogy of Exponential Service Times with Poisson Arrivals

It has been proved in **sec. 23.7-3** that if number of arrivals (n) follows the *Poisson distribution,* then the inter-arrival time (T) will follow the *exponential one,* and *vice-versa.*

In the like manner, it can also be shown that, if the time (t) to complete the service of a unit follows the *exponential distribution* given by the probability density function

$$s(t) = \mu e^{-\mu t} \qquad \text{...(23.45)}$$

where μ is the mean servicing rate for a particular station, then the number (n) of departures in time T (if there were no enforced idle time) will follow the *Poisson distribution* given by

$$\varphi_T(n) = \text{Prob. } [n \text{ services in time } T, \text{ if servicing is going on throughout } T] = (\mu T)^n e^{-\mu T}/n\,! \qquad \text{...(23.46)}$$

Consequently, from (23.27), it can be shown that

$$\varphi_{\Delta T}(0) = \text{Prob. } [\textit{no service in } \Delta T] = 1 - \mu \Delta T \qquad \text{...(23.47)}$$

and

$$\varphi_{\Delta T}(1) = \text{Prob. } [\textit{one service in } \Delta T] = \mu \Delta T. \qquad \text{...(23.48)}$$

23.7-8. Erlang Service Time Distribution (E_k).

So far it is considered (in **23.7-3** and **23.7-6**) and seen that the inter-arrival time distribution and service time distribution both will follow the exponential assumptions given by

$$a(T) = \lambda e^{-\lambda t}, \text{ and } s(t) = \mu e^{-\mu t}, \text{ respectively.} \qquad \text{...(23.49)}$$

These only give a one particular family of possible arrival and service time distribution, respectively.

A two parameter (μ and k) generalisation of the exponential family, which is of great importance in queueing problems is called the *Erlang* family of service time distribution (named for *A.K. Erlang,* the ***Danish*** telephone engineer. This is defined by its probability density function,

$$s(t, \mu, k) = (k\mu)^k t^{k-1} e^{-k\mu t}/(k-1)\,! = C_k t^{k-1} e^{-k\mu t} \qquad \text{...(23.50)}$$

where $C_k = (k\mu)^k/(k-1)\,!$, $0 \leq t < \infty$, $k \geq 1$.

It should be noted carefully that (23.50) gives us the exponential distribution given by (23.49) for $k = 1$.

Let $t_1, t_2, t_3, ..., t_k$ be the servicing time for any customer in respective k phases, then the total service time t is given by

$$t = t_1 + t_2 + \ldots + t_k.$$

Also, each of the times $t_1, t_2, ..., t_k$ is independently and exponentially distributed with parameter $k.\mu$.

Hence, $P\,[t \leq t_1 + t_2 \ldots + t_k \leq t + \Delta t]$

$$= \iint \ldots \int p(t_1)\, p(t_2) \ldots p(t_k)\, dt_1\, dt_2 \ldots dt_k. \text{ for } t \leq t_i \leq t + \Delta t, i = 1, 2, ..., k.$$

$$= \iint \ldots \int (k\mu e^{-k\mu t_1} \ldots (k\mu e^{-k\mu t_k})\, dt_1 \ldots dt_k. \quad [\text{since } p(t_i) = k\mu e^{-k\mu t_i}]$$

$$= (k\mu)^k \iint \ldots \int e^{-k\mu \sum_{i=1}^{k} t_i}\, dt_1 \ldots dt_k.$$

Now applying ***Drichlet's*** theorem of multiple integrals,

$$= (k\,\mu)^k \frac{\Gamma 1^k}{\Gamma(k)} e^{-k\mu t} t^{k-1}$$

$$= \frac{(k\,\mu)^k}{\Gamma(k)} t^{k-1} e^{-k\mu t}, k \geq 0.$$

Note. The distribution is a modified χ^2 distribution with mean $(1/\mu)$ and $2k$ degrees of freedom.

Thus, if we have service times $t_1, t_2, t_3, ..., t_k$ in k phases which are exponentially distributed variables with a common mean $1/k\mu$, then $t = t_1 + t_2 + t_3 + ... + t_k$ has the *Erlangian (Gamma)* distribution with k phases and parameter μ.

Derivation of Erlangion Service Distribution

In the *Fig.* 23.12 for *Erlangian service time distribution* for $k = 3$ phases, it is observed that —

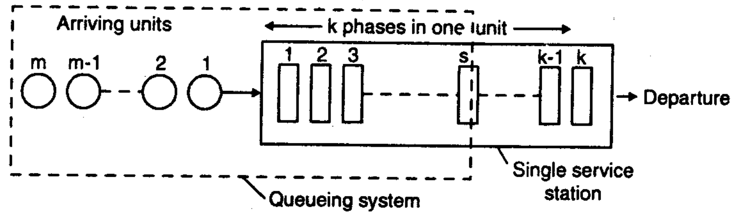


Fig. 23.12

(*i*) each phase of service is exponential;

(*ii*) a unit enters phase 3 first, then goes to 2, to 1 and out;

(*iii*) no other unit can enter phase 3 until the previous unit leaves phase 1.

Properties. The *Erlang* family of service time distribution has many interesting properties such as :

(1) All the members share the common mean $1/\mu$, that is, $E(t) = 1/K\mu$, and variance of t is given by $V(t) = 1/k\mu^2$

(2) One parameter family is obtained by setting $k = 1$.

(3) The mode is located at :

$t = 0$ for $k = 1$,
$t = 1/2\mu$ for $k = 2$,

$t = (k-1)/k\mu$ for general k,
$t = 1/\mu$ for $k \to \infty$

(4) As $k \to \infty$, $V(t) \to 0$, [since $V(t) = 1/k\mu^2$]

(5) For constant service time, $k \to \infty$ **[Note.]**

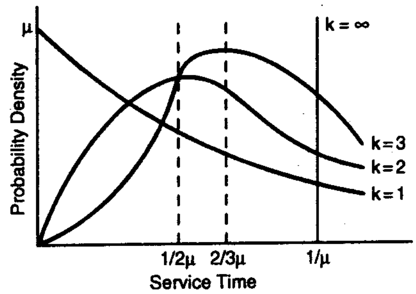


Fig. 23.13. The Earlang family of service time distribution.

23.8. SOME QUALITATIVE ASSUMPTIONS

It is essential to pay much attention to physical significance of following three qualitative assumptions for further discussion of queueing system.

(a) **Stationary waiting line.** The probability that n customers arrive in a time interval $(T, T+t)$ is independent of T and is a function of the variables n and t both.

(b) **Absence of after effects.** This means that the probability of n customers arriving during a time interval $(T, T+t)$ does not depend on the number of customers arriving before T.

(c) **The orderlines of the waiting line.** It expresses the practical impossibility of two or more customers arriving at the same instant of time.

A waiting line (queue) satisfying the above three conditions is usually called a *Simple Queue*.

Q. 1. (*a*) Explain the basic queueing process. What are the important random variates in queueing system to be investigated ?

(*b*) What do you understand by (*i*) queue discipline (*ii*) input, and (*iii*) holding time ?

2. Explain : (*i*) the constituents of a queueing model, and (*ii*) the characteristics to be analysed.
3. State some of the important distributions of arrival intervals and service times. **[Delhi MA/M.Sc (OR) 93]**
4. Give the essential characteristics of the queueing process.
5. State some of the important inter-arrival and service time distributions.
6. What do you understand by a queue ? Give some important applications of queueing theory.
7. Find the mode of the *k*-Erlang distribution with parameter μ.
8. What do you understand by a optimum service rate ? Show how some important queueing formulae are used in determining the optimum service rate and the number of channels.
9. Prove that, for the Erlang distribution with parameters μ and k, the mode is at $(1 - 1/k)\, 1/\mu$, the mean is $1/\mu$, and the variance is $1/k\mu^2$. **[Kanpur 96; Garhwal M.Sc. (Stat) 92; Meerut (M.Sc. Maths) 90]**
10. Write a note on *Erlangian* distribution. **[Meerut (OR) 2003; Delhi M.A/M.Sc. (OR.) 90]**

23.9. KENDALL'S NOTATION FOR REPRESENTING QUEUEING MODELS

Generally queueing model may be completely specified in the following symbolic form : $(a \mid b \mid c) : (d \mid e)$, where

$a \equiv$ probability law for the arrival (or inter-arrival) time.

$b \equiv$ probability law according to which the customers are being served,

$c \equiv$ number of channels (or service stations),

$d \equiv$ capacity of the system, *i.e.*, the maximum number allowed in the system (in service and waiting),

$e \equiv$ queue discipline.

It is important to note that first three characteristics $(a \mid b \mid c)$ in the above notation were introduced by *D. Kendall* (1953). Later, *A. Lee* (1966) added the *fourth* (d) and the *fifth* (e) characteristics to the notation. Although, it is noticed that this notation is not suitable for describing complex models such as queues in series, or network queues. This will be suitable, however, for the purpose of material presented here, and the reader should find it helpful in comparing the different models.

Q. Explain Kendall's notations for representing queueing models. **[JNTU (Mech. & Prod.) 2004]**

23.10. CLASSIFICATION OF QUEUEING MODELS

For simplicity, the queueing models presented here are classified as follows :

I—Probabilistic Queueing Models

Model I. (*Erlang Model*). This model is symbolically represented by $(M \mid M \mid 1) : (\infty \mid FCFS)$. This denotes *Poisson* arrival (exponential inter-arrival), *Poisson* departure (exponential service time), single server, infinite capacity and *"First Come, First Served"* service discipline.

Note. Since the *'Poisson'* and the *'Exponential'* distributions are related to each other (see sec. 23.7-3), both of them are denoted by the same letter '*M*'. Letter '*M*' is used due to *Markovian property* of exponential process.

Model II. (*General Erlang Model*). Although this model is also represented by $(M \mid M \mid 1) : (\infty \mid FCFS)$, but this is a general queueing model in which the rate of arrival and the service depend on the length n of the line.

Model III. This model is represented by $(M \mid M \mid 1) : (N \mid FCFS)$. In this model, capacity of the system is limited (finite), say N. Obviously, the number of arrivals will not exceed the number N in any case.

Model IV. This model is represented by $(M \mid M \mid s) : (\infty \mid FCFS)$, in which the number of service stations is s in parallel.

Model V. This model is represented by $(M \mid E_k \mid 1) : (\infty \mid FCFS)$, that is, *Poisson* arrivals, *Erlangian* service time for k phases (see **sec. 23.7-8**), and a single server.

Model VI. (***Machine Repairing Model***). This model is represented by $(M \mid M \mid R) : (K \mid GD)$, $K > R$, that is, *Poisson* arrivals, *Exponential* service time, R repairmen, and K machines in the system, and general queue discipline.

Model VII. ***Power-Supply Model.***

Model VIII. ***Economic Cost Profit Models.***

Model IX. $(M \mid G \mid 1) : (\infty \mid GD)$, where G is the general output distribution, and GD represents a general service discipline.

II—Mixed Queueing Model

Model X. $(M \mid D \mid 1) : (\infty \mid FCFS)$, where D stands for deterministic service time.

III—Deterministic Queueing Model

Model XI. $(D \mid D \mid 1) : (K - 1 \mid FCFS)$, where

$D \rightarrow$ Deterministic arrivals, *i.e.*, inter-arrival time distribution is *constant* or *regular*.

$D \rightarrow$ Deterministic service time distribution.

Q. 1. Give a brief summery of the various types of queueing models. **[Bhubnashwar (IT) 2004]**

2. Write a note on *Kendal* and *Lee's* notation for the identification of queues. **[Karnataka 94]**

23.11. SOLUTION OF QUEUEING MODELS AND LIMITATIONS FOR ITS APPLICATIONS

The solution of queueing models as classified in **sec. 23.10** will consist of the following parts :

(*a*) ***To obtain the system of steady state equations governing the queue.***

(*b*) ***To solve these equations for finding out the probability distribution of queue length.***

(*c*) ***To obtain probability density function for waiting time distribution.***

(*d*) ***To find the busy period distribution.***

(*e*) ***To derive formula for*** L_s, L_q, $(L \mid L > 0)$, W_s, W_q, $(W \mid W > 0)$, ***and*** $Var\{n\}$, ***etc.***

(*f*) ***Also, to obtain the probability of arrival during the service time of any customer.***

The analytic procedure may be adopted for solving the steady state equations for ***Models*** I-IV. Since the analytic procedure seems to be more complicated for ***Model*** V, so we shall adopt the increasingly powerful technique of *Generating Functions.*

Limitation for Application of Queueing Model :

The single channel queueing model can be fitted in situations where the following conditions are satisfied.

(i) The number of arrivals rate is denoted by λ.

(ii) The service time has exponential distribution. The average service rate is denoted by μ.

(iii) Arrivals are from infinite population.

(iv) The queue discipline is *FCFS (i.e. FIFO), i.e.* the customers are served on a first come first served basis—

(v) There is only a single service station.

(vi) The mean arrival rate is less than the mean service rate, *i.e.* $\lambda < \mu$.

(vii)The waiting space available for customers in the queue is infinite.

The single channel queueing model is the most simple model which is based on the above mentioned assumption. But, in reality, there are several limitations of this modes in its applications. One abvious limitation is the possibility that the waiting space, in fact, be limited. Other possibility is that arrival rate is state dependent. That is, potential customers are discouraged from entering the queue if they observe a long line at the time they arrive. Another practical limitation of the model is that the arrival process is not stationary. It is quite possible that the service station would experience peak period, and slack periods during which the arrival rate is higher and lower respectively than the over all average. These could occur at particular times during a day or a week or particular weeks during a year. The population of customers served may be finite, the queue discipline may not be *first come first served.* In general, the validity of these models depends on the assumptions that are often unrealistic in practice.

Even when the assumptions are realistic, there is another limitation of queueing theory that is often overlooked. Queueing models give steady state solution, *i.e.* the model tells us what will happen after queueing system hes been in operation long enough to eliminate the effects of starting with an empty queue at the beginning of each business day. In some applications, the queueing system never reaches a steady state, so the model solutions are of little importance.

Q. 1. Mention any seven conditions that must be fulfilled by the situations if they were to be described by a queueing model. What are the limitations of this model in its applications. **[C.A. (Nov.) 93]**

2. (a) Describe Queueing model and its significance. What are various queue models, give in details ?
(b) List the factors that constitute the basic elements of queuing model. For each of those, enumerate the alternatives possible. Represent them diagramatically to cover all possible implimentations of a queueing model ? **[IGNOU 99, 98, 96]**

We now proceed to discuss each model in detail with the help of various interesting examples.

23.12. MODEL I. (*M* | *M* | 1) : (∞ | *FCFS*) : BIRTH AND DEATH MODEL

This model is also called the ***'birth and death model'***.

I. To obtain the system of steady-state equations. **[Agra 99; Meerut 93]**

The probability that there will be n units ($n > 0$) in the system at time $(t + \Delta t)$ may be expressed as the sum of three independent compound probabilities, by using the fundamental properties of probability, Poisson arrivals, and of exponential service times.

(*i*) ***The product of three probabilities*** (see ***Fig.* 23.14**),

(*a*) that there are n units in the system at time $t = P_n(t)$

(*b*) that there is no arrival in time $\Delta t = P_0(\Delta t) = 1 - \lambda \Delta t$ [see (23.20)]

(*c*) that there is no service in time $\Delta t = \varphi_{\Delta t}(0) = 1 - \mu \Delta t$; [see (23.47)]

is given by

$$P_n(t).(1 - \lambda \Delta t).(1 - \mu \Delta t) \cong P_n(t)\,[1 - (\lambda + \mu)\,\Delta t] + O_1(\Delta t).$$

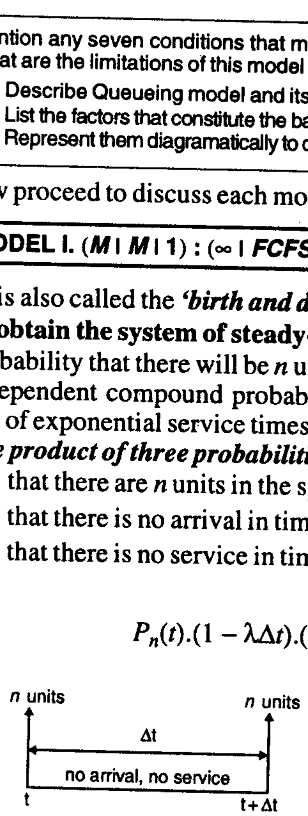


Fig. 23.14.

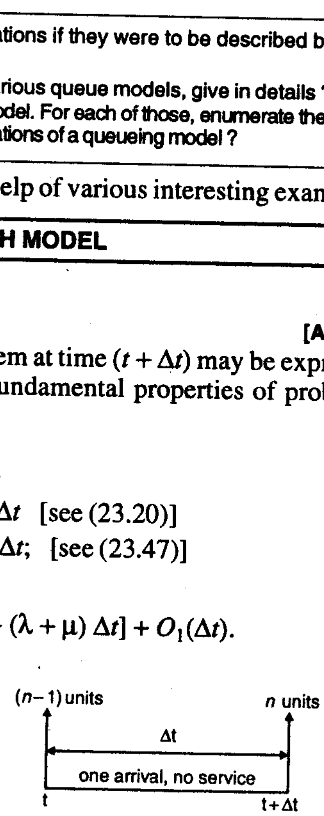


Fig. 23.15

(*ii*) ***The product of three probabilities*** (see ***Fig.* 23.15**),

(*a*) that there are $(n - 1)$ units in the system at time $t = P_{n-1}(t)$;

(*b*) that there is one arrival in time $\Delta t = P_1(\Delta t) = \lambda \Delta t$ [see (23.21)]

(*c*) that there is no service in $\Delta t = \varphi_{\Delta t}(0) = 1 - \mu \Delta t$;

is given by $\quad P_{n-1}(t).(\lambda \Delta t).(1 - \mu \Delta t) \cong \lambda P_{n-1}(t)\,\Delta t + O_2(\Delta t)$,

(*iii*) ***The product of probabilities*** (see ***Fig.* 23.16**),

(*a*) that there are $(n + 1)$ units in the system at time $t = P_{n+1}(t)$

(*b*) that there is no arrival in time Δt
$= P_0(\Delta t) = 1 - \lambda \Delta t$

(*c*) that there is one service in time
$\Delta t = \varphi_{\Delta t}(1) = \mu \Delta t$; [see (23.48)]

is given by

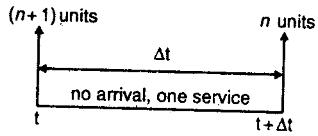


Fig. 23.16

$$P_{n+1}(t)\,(1 - \lambda \Delta t)\,\mu \Delta t \cong P_{n+1}(t)\,\mu \Delta t + O_3(\Delta t).$$

Note. The probabilities of more than one unit arriving and/or being served during the interval Δt are assumed to be negligible. Further, $O_1(\Delta t)$, $O_2(\Delta t)$, $O_3(\Delta t)$ are also the functions of Δt in the sense of notation '$O(\Delta t)$' as explained in sec. 23.7-2.

Now, by adding above three independent compound probabilities, we obtain the probability of n units in the system at time $(t + \Delta t)$, *i.e.*,

$$P_n(t + \Delta t) = P_n(t)\,[1 - (\lambda + \mu)\,\Delta t] + P_{n-1}(t)\,\lambda \Delta t + P_{n+1}(t)\,\mu \Delta t + O(\Delta t), \quad \ldots(23.51)$$

where $O(\Delta t) = O_1(\Delta t) + O_2(\Delta t) + O_3(\Delta t)$.

The equation (23.51) may be written as

$$\frac{P_n(t+\Delta t)-P_n(t)}{\Delta t}=-(\lambda+\mu)\,P_n(t)+\lambda P_{n-1}(t)+\mu P_{n+1}(t)+\frac{O(\Delta t)}{\Delta t}$$

Now, taking limit as $\Delta t \to 0$ on both sides,

$$\lim_{\Delta t\to 0}\frac{P_n(t+\Delta t)-P_n(t)}{\Delta t}=\lim_{\Delta t\to 0}\left[-(\lambda+\mu)\,P_n(t)+\lambda P_{n-1}(t)+\mu P_{n+1}(t)+\frac{O(\Delta t)}{\Delta t}\right]$$

or

$$\frac{dP_n(t)}{dt}=-(\lambda+\mu)\,P_n(t)+\lambda P_{n-1}(t)+\mu P_{n+1}(t);\; n>0\left(\text{since }\lim_{\Delta t\to 0}\frac{O(\Delta t)}{\Delta t}=0\right) \quad ...(23.52)$$

In a similar fashion, the probability that there will be no unit (*i.e.* $n = 0$) in the system at time $(t + \Delta t)$ will be the sum of the following two independent probabilities :

(*i*) Prob. [*that there is no unit in the system at time t and no arrival in time* Δt] $= P_0(t).(1-\lambda\Delta t)$.
question of any service in time Δt does not arise because there are *no* units in the system at time *t;* and

(*ii*) Prob. [*that there is one unit in the system at time t, one unit serviced in* Δt, *and no arrival in* Δt]

$$= P_1(t).\mu\Delta t.(1-\lambda\Delta t) \cong P_1(t)\mu\Delta t + O(\Delta t)$$

Now, adding these two probabilities, we get

$$P_0(t+\Delta t)=P_0(t)\,[1-\lambda\Delta t]+P_1(t)\,\mu\Delta t+O(\Delta t) \quad ...(23.53)$$

or

$$\frac{P_0(t+\Delta t)-P_0(t)}{\Delta t}=-\lambda P_0(t)+\mu P_1(t)+\frac{O(\Delta t)}{\Delta t}.$$

Now, taking limit on both sides as $\Delta t \to 0$

$$\frac{dP_0(t)}{dt}=-\lambda P_0(t)+\mu P_1(t),\ \text{for } n=0 \quad ...(23.54)$$

Since only the steady state probabilities are considered here (see **sec. 23.4**),

$$\lim_{t\to\infty}\frac{d\,[P_n(t)]}{dt}=0,\ \text{for } n\ge 0 \text{ and } \lim_{t\to\infty}P_n(t)=P_n \text{ (which is independent of } t)$$

Consequently, the equations (23.52) and (23.54) can be written as :

$$0=-(\lambda+\mu)\,P_n+\lambda P_{n-1}+\mu P_{n+1},\quad \text{if } n>0 \quad ...(23.52a)$$

$$0=-\lambda P_0+\mu P_1,\quad \text{if } n=0 \quad ...(23.54a)$$

In this way, the equations (23.52a) and (23.54a) constitute the *system of steady state difference equations* for this model.

Q. In a single server, Poisson arrival and exponential service time queueing system, show that probability P_n of n customers in steady state satisfy the following equations :

$\lambda P_0=\mu P_1$, $(\lambda+\mu)P_1=\mu P_2+\lambda P_0$, and $(\lambda+\mu)P_n=\mu P_{n+1}+\lambda P_{n-1}$, for $n\ge 2$.

II. To solve the system of difference equations. **[Meerut 97 P]**

By the technique of successive substitution. we solve the difference equations :

$$0=-(\lambda+\mu)\,P_n+\lambda P_{n-1}+\mu P_{n+1},\quad \text{if } n>0,$$

$$0=-\lambda P_0+\mu P_1,\quad \text{if } n=0,$$

Since $P_0 = P_0$

$P_1=\frac{\lambda}{\mu}P_0$ [from the equation (23.54a) for $n = 0$]

$P_2=\frac{\lambda}{\mu}P_1=\left(\frac{\lambda}{\mu}\right)^2 P_0$ [from letting $n = 1$ in the eqn. (23.52*a*) for $n > 0$ and substituting for P_1]

$P_3=\frac{\lambda}{\mu}P_2=\left(\frac{\lambda}{\mu}\right)^3 P_0$ [from letting $n = 2$ in eqn. (23.52*a*) for $n > 0$ and substituting for P_2]

$\vdots \quad \vdots \quad \vdots$

$P_n=\frac{\lambda}{\mu}P_{n-1}=\left(\frac{\lambda}{\mu}\right)^n P_0$ [for $n \ge 0$] ...(23.55)

Now using the fact that $\sum_{n=0}^{\infty} P_n = 1$,

$$P_0 + \frac{\lambda}{\mu} P_0 + \left(\frac{\lambda}{\mu}\right)^2 P_0 + \ldots + \left(\frac{\lambda}{\mu}\right)^n P_0 + \ldots = 1 \quad \text{or} \quad P_0\left[1 + \frac{\lambda}{\mu} + \left(\frac{\lambda}{\mu}\right)^2 \ldots\right] = 1$$

or $$P_0\left[\frac{1}{1 - \lambda/\mu}\right] = 1 \quad \text{[since } (\lambda/\mu) < 1 \text{ as explained in } \textbf{sec. 23.6}\text{, sum of infinite G.P. is valid]}$$

or $$\boldsymbol{P_0 = 1 - (\lambda/\mu).} \qquad \text{...(23.56)}$$

Now, substituting the value of P_0 from (23.56) in (23.55), we get

$$\boldsymbol{P_n = \left(\frac{\lambda}{\mu}\right)^n \left(1 - \frac{\lambda}{\mu}\right) = \rho^n(1 - \rho), \; \left(\rho = \frac{\lambda}{\mu} < 1, n \geq 0\right)} \qquad \text{...(23.57)}$$

Thus, equations (23.56) and (23.57) rather give the required *probability distribution of queue length.*

Q. 1. Derive the differential-difference equations for the queueing model $(M \mid M \mid 1) : (\infty \mid FCFS)$. How would you proceed to solve the model? **[Meerut (Stat.) 98; Delhi M.A/M.Sc. (OR). 90]**

2. Obtain the steady state solution of (M | M | 1) : (∞ | FCFS) system and also find expected value of queue length n. **[Meerut (Maths.) 97P; Garhwal (Stat.) 96]**

3. Explain (M | M | 1) : (∞ | FCFS) queueing model, derive and solve the difference equations in steady state, of the model. **[Agra 93; Garhwal M.Sc. (Stat.) 93]**

4. Show that for a single service station, Poisson arrivals and exponential service time, the probability that exactly n calling units are in the queueing system is $P_n = (1 - \rho)\, \rho^n$, $n \geq 0$, where ρ is the traffic intensity.

Further, we may also compute a useful probability, *viz.*,

$$\text{Prob. [}\textit{queue size} \geq N\text{]} = \sum_{n=N}^{\infty} P_n = \sum_{n=0}^{\infty} P_n - \sum_{n=0}^{N-1} P_n = 1 - (P_0 + P_1 + \ldots + P_{N-1})$$

$$= 1 - \left[P_0 + \frac{\lambda}{\mu} P_0 + \ldots + \left(\frac{\lambda}{\mu}\right)^{N-1} P_0\right] = 1 - P_0\left[\frac{1 - (\lambda/\mu)^N}{1 - \lambda/\mu}\right]$$

$$= 1 - \left(1 - \frac{\lambda}{\mu}\right)\left[\frac{1 - (\lambda/\mu)^N}{1 - \lambda/\mu}\right] = \left(\frac{\lambda}{\mu}\right)^N \qquad \text{[from (5.56), } P_0 = 1 - (\lambda/\mu)\text{]}$$

$\therefore$ $$\textbf{Prob. [}\boldsymbol{\textit{queue size} \geq N}\textbf{]} \boldsymbol{= (\lambda/\mu)^N = \rho^N.} \qquad \text{...(23.58)}$$

Q. 1. Describe a queue model and steady state equations of $M \mid M \mid 1$ queues. What is the prob. that at least one unit is present in the system. **[Meerut (I.P.M.) 90]**

2. Explain M | M | 1 queue model in the transient state. Derive steady state solution for the M | M | 1 queue model. **[Garhwal M.Sc. (Math.) 91]**

3. If P_n represents the probability of finding n in the long run in a queueing system with Poisson arrivals having parameter λ and exponential service times with parameter μ, show that.

$$\lambda P_{n-1} - (\lambda + \mu) P_n + \mu P_{n+1} = 0 \quad \text{for } n > 0$$

and $$-\lambda P_0 + \mu P_1 = 0. \quad \text{for } n = 0$$

Solve these difference equations and obtain P_n in terms of $P = \lambda/\mu$. **[I.A.S. (Main) 95]**

III. To obtain probability density function of waiting time (excluding service time) distribution.

[Kanpur 93]

In the steady state, each customer has the same waiting time distribution. This is a continuous distribution with probability density function $\Psi(w)$, and we denote by $\Psi(w)\, dw$ the probability that a customer begins to be served in the interval $(w, w + dw)$, where w is measured from the time of his arrival. We suppose that a customer arrives at

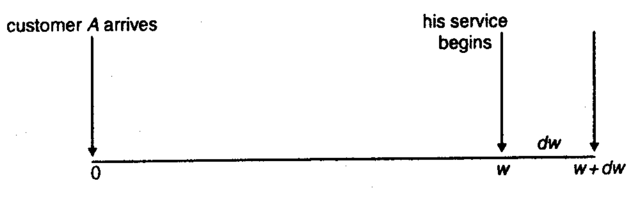


Fig. 23.17

time $w = 0$ and service begins in the interval $(w, w + dw)$. *Fig.* 23.17 illustrates this situation. For convenience, we label the customer as *A*.

(*i*) There is a finite probability that waiting time is zero (P_0 the probability that the system is empty).

(*ii*) If there are *n* customers already in the system when the customer *A arrives, n* must leave before the service of *A* begins. More precisely, $(n - 1)$ customers must leave during the time interval $(0, w)$, and the *n*th customer during $(w, w + dw)$.

[If n customers left by the time w, service of A could begin before the interval (w, w + dw); and if fewer than (n − 1) left by time w, service could only begin in (w, w + dw); if there were two or more departures in that interval, the probability is O (dt) which may be ignored].

The server's mean rate of service is μ in unit time, or μw in time w, and the probability of $(n - 1)$ departures in time w, during which the sever is busy, is the appropriate term of the Poisson distribution $(\mu w)^{n-1} e^{-\mu w}/(n-1)!$.

Let there be *n* units in the system (see *Fig.* 23.18), then

$\Psi_n(w)\, dw =$ Prob. [$(n - 1)$ *units are served at time w*] $\times$ Prob. [*one unit is served in time dw*],

or
$$\Psi_n(w)\, dw = \frac{(\mu w)^{n-1} e^{-\mu w}}{(n-1)!} \times \mu\, dw. \qquad \text{...(23.59)}$$

Let W be the waiting time of a unit who has to wait such that $w \le W \le w\, dw$, then the probability $\Psi(w)\, dw$ is given by

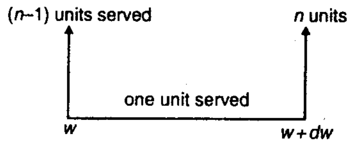


Fig. 23.18

$\Psi(w)\, dw =$ Prob. $(w \le W \le w + dw)$

= (The probability of *n* customers in the system when customer *A* arrives) $\times$ [the probability that exactly $n - 1$ customers leave in $(0, w)$] $\times$ [the probability that *n*th customer leaves in $(w, w + dw)$], summed over all *n* from 1 to ∞

$$= \sum_{n=1}^{\infty} P_n\, \Psi_n(w)\, dw = \sum_{n=1}^{\infty} \left(\frac{\lambda}{\mu}\right)^n \left(1 - \frac{\lambda}{\mu}\right) \cdot \frac{(\mu w)^{n-1} e^{-\mu w}}{(n-1)!}\, \mu\, dw$$

[from (23.57) and (23.59)]

$$P_n = \left(\frac{\lambda}{\mu}\right)^n \left(1 - \frac{\lambda}{\mu}\right) = \frac{\lambda}{\mu}\left(1 - \frac{\lambda}{\mu}\right) \mu\, e^{-\mu w} \sum_{n=1}^{\infty} \frac{[(\lambda/\mu)\,(\mu w)]^{n-1}}{(n-1)!}\, dw = \lambda \left(1 - \frac{\lambda}{\mu}\right) e^{-\mu w} \sum_{n=1}^{\infty} \frac{(\lambda w)^{n-1}}{(n-1)!}\, dw$$

$$= \lambda \left(1 - \frac{\lambda}{\mu}\right) e^{-\mu w} \left[1 + \frac{(\lambda w)}{1!} + \frac{(\lambda w)^2}{2!} + \ldots\right] dw \qquad = \lambda \left(1 - \frac{\lambda}{\mu}\right) e^{-\mu w} e^{\lambda w}\, dw$$

$$\therefore \qquad \boldsymbol{\Psi(w) = \lambda \left(1 - \frac{\lambda}{\mu}\right) e^{-(\mu - \lambda) w}, \quad w > 0.} \qquad \text{...(23.60)}$$

This result may also be obtained by using the *Laplace transform* of service time distribution, and the properties of the waiting time distribution may be found from its *Laplace transform.*

Obviously, $\int_0^{\infty} \Psi(w)\, dw \ne 1$, because it has the value λ/μ.

It is important to note that the case for which $w = 0$ has been excluded in eqn. (23.60). Thus,

Prob. $[W = 0]$ = Prob. [*no unit in the system*] $= P_0 = 1 - (\lambda/\mu)$. [from eqn. (23.56)]

Now, the sum of all probabilities of waiting time

$$= \int_0^{\infty} \text{Prob. } [w \le W \le w + dw] + \text{Prob. } [W = 0] \qquad \text{...(23.61)}$$

$$= \int_0^{\infty} \lambda \left(1 - \frac{\lambda}{\mu}\right) e^{-(\mu - \lambda) w}\, dw + \left(1 - \frac{\lambda}{\mu}\right)$$

$$= \frac{\lambda}{\mu} + \left(1 - \frac{\lambda}{\mu}\right) = 1$$

Hence it is concluded that the complete distribution for waiting time is partly continuous and partly discrete :

(*i*) ***continuous*** for $w \le W \le w + dw$ with probability density function $\Psi(w)$ given by eqn. (23.60); and (*ii*) ***discrete*** for $W = 0$, with Prob $(W = 0) = 1 - (\lambda/\mu)$.

The probability that waiting time exceeds w is given by

$$\int_w^{\infty} \Psi(w)\, dw = \int_w^{\infty} \lambda\left(1 - \frac{\lambda}{\mu}\right) e^{-(\mu-\lambda)w}\, dw = \left(-\frac{\lambda}{\mu} e^{-(\mu-\lambda)w}\right)_w^{\infty} = \frac{\lambda}{\mu} e^{-(\mu-\lambda)w} = \rho e^{-(\mu-\lambda)w}$$

which does not include the service time.

Q. 1. Define cumulative probability distribution of waiting time for a customer who has to wait and show that in an $(M \mid M \mid 1) : (\infty \mid FIFO)$ queue system, it is given by $1 - \rho\, e^{-\mu t(1-\rho)}$ where $\rho = \lambda/\mu$.

[**Hint.** Cum. Distribution $= \int_0^t \lambda(1-\rho)\, e^{-\mu(1-\rho)w}\, dw = (1-\rho)$.]

2. Define (M I M I 1) system. **[IGNOU 99 (Dec.)]**

IV. To find prob. distribution of time spent in the system (busy period distribution).

[Kanpur M.Sc. (Math.) 93]

In order to find the probability density function for the distribution of total time (waiting + service) an arrival spends in the system, let $\Psi(w \mid w > 0)$ = probability density function for waiting time such that *a person has to wait.*

The statement *"person has to wait"* is meant that the server remains busy in the *busy period.*

Applying the rule of conditional probability,

$$\Psi(w \mid w > 0)\, dw = \frac{\Psi(w)\, dw}{\text{Prob. } (w > 0)} = \frac{\Psi(w)\, dw}{\int_0^{\infty} \Psi(w)\, dw}.$$

Substituting the value for $\Psi(w)$ from eqn. (23.60),

$$\Psi(w \mid w > 0)\, dw = \frac{\lambda(1 - \lambda/\mu)\, e^{-(\mu-\lambda)w}\, dw}{\int_0^{\infty} \lambda(1-\lambda/\mu)\, e^{-(\mu-\lambda)w}\, dw} = \frac{\lambda(1-\lambda/\mu)\, e^{-(\mu-\lambda)w}\, dw}{\lambda/\mu}$$

or $$\Psi(w \mid w > 0) = (\mu - \lambda)\, e^{-(\mu-\lambda)w} \qquad \text{...(23.62)}$$

Here, $\int_0^{\infty} \Psi(w \mid w > 0)\, dw = \int_0^{\infty} (\mu - \lambda)\, e^{-(\mu-\lambda)w}\, dw = 1.$

Hence it gives the required probability density function for the busy period.

If service time is included then,

$$P(W \geq w) = \int_w^{\infty} (\mu - \lambda)\, e^{-(\mu-\lambda)w}\, dw = e^{-(\mu-\lambda)w}$$

V. MEASURES OF MODEL I :

(*i*) To find expected (average) number of units in the system, L_s.

By definition of expected value,

$$L_s = \sum_{n=1}^{\infty} n P_n = \sum_{n=1}^{\infty} n\left(\frac{\lambda}{\mu}\right)^n\left(1 - \frac{\lambda}{\mu}\right) = \left(1 - \frac{\lambda}{\mu}\right)\frac{\lambda}{\mu}\sum_{n=1}^{\infty} n\left(\frac{\lambda}{\mu}\right)^{n-1}$$

$$= \left(1 - \frac{\lambda}{\mu}\right)\frac{\lambda}{\mu}\left[1 + 2\left(\frac{\lambda}{\mu}\right) + 3\left(\frac{\lambda}{\mu}\right)^2 + \dots \infty\right] = \left(1 - \frac{\lambda}{\mu}\right)\frac{\lambda}{\mu}\left[\frac{1}{(1-\lambda/\mu)^2}\right]^* \text{ (see foot-note)}$$

---

* Let $S = 1 + 2\left(\frac{\lambda}{\mu}\right) + 3\left(\frac{\lambda}{\mu}\right)^2 + \dots \infty$, which is Arithmatico-Geometric series,

$\therefore \quad \frac{\lambda}{\mu} S = \left(\frac{\lambda}{\mu}\right) + 2\left(\frac{\lambda}{\mu}\right)^2 + \dots \infty$

On subtracting,

$\left(1 - \frac{\lambda}{\mu}\right) S = 1 + \left(\frac{\lambda}{\mu}\right) + \left(\frac{\lambda}{\mu}\right)^2 + \dots \infty = \frac{1}{1 - \lambda/\mu}$ (sum of infinite G.P.)

$\therefore \quad S = \frac{1}{(1-\lambda/\mu)^2}.$

or $$L_s = \frac{\lambda/\mu}{(1-\lambda/\mu)} = \frac{\rho}{1-\rho}, \text{ where } \rho = \lambda/\mu < 1 \quad ...(23.63)$$

which is the required formula.

Q. 1. In a certain queueing system with one server, the arrivals obey a Poisson distribution with mean λ and the service time distribution has mean 1/μ. Obtain the generating function of the length of the queue which a departing customer leaves behind him.

2. Show that for a single service station, Poisson arrivals and exponential service time, the probability that exactly *an calling units are in queueing system is* $P_n = (l-\rho)\rho^n, n \geq 0$ (*ρ is the traffic intensity).*
Also, find the expected line length. **[B.H.U. 93]**

3. Show that average number of units in a *M* I *M* I 1 system is equal to ρ/(1 – ρ). **[Agra 99; Raj. Univ. (M. Phil) 93]**

4. Discuss (M I M I 1) : (∞ I FCFS) queueing model and find the expected line length $E(L_s)$ in the system. **[Garhwal M.Sc. (Math.) 96]**

5. For the M I M I 1 queueing system, find :
(a) Expected value of queue length *n*
(b) Prob. distribution of waiting time *w*. **[Meerut M.Sc. (Math.) BP-96]**

6. Explain a system with Poisson input, exponential waiting time with single chennel. Also determining the average length of the waiting time. **[Meerut 2002]**

(*ii*) To find expected (average) queue length, L_q : **[IGNOU 1999; Meerut 97 P, 93; Kanpur 93]**

Since there are (*n* – 1) units in the queue excluding one being serviced,

$$L_q = \sum_{n=1}^{\infty}(n-1)P_n = \sum_{n=1}^{\infty} nP_n - \sum_{n=1}^{\infty} P_n = \sum_{n=1}^{\infty} nP_n - \left[\sum_{n=0}^{\infty} P_n - P_0\right] = L_s - [1-P_0] \quad \left(\text{since } \sum_{n=0}^{\infty} P_n = 1\right)$$

Substituting the value of P_0 from (23.56), we have

$$L_q = L_s - 1 + \left(1 - \frac{\lambda}{\mu}\right) \quad ...(23.64)$$

or $$L_q = L_s - \frac{\lambda}{\mu} = \frac{\rho^2}{1-\rho}, \text{ where } L_s = \frac{\lambda/\mu}{1-\lambda/\mu} = \frac{\rho}{1-\rho}.$$

Q. 1. For *M* I *M* I 1 model, obtain the expected number of waiting customers if the queueing process is going on for a long time.

2. Describe the system of steady state equations for a queueing model (*M* I *M* I 1) : (*FCFS*, ∞) and obtain their solution. Obtain the mean queue length and mean number of units in the system. **[Meerut 95, 90]**

(*iii*) To find mean (or expected) waiting time in the queue (excluding service time), W_q :

Since expected time an arrival spends in the queue is given by

$$W_q = \int_0^{\infty} w\Psi(w)\,dw = \int_0^{\infty} w.\lambda\left(1-\frac{\lambda}{\mu}\right)e^{-(\mu-\lambda)w}\,dw \quad \text{[from eqn. (23.60)]}$$

Integrating by parts,

$$= \lambda\left(1-\frac{\lambda}{\mu}\right)\left[w \cdot \frac{e^{-(\mu-\lambda)w}}{-(\mu-\lambda)} - \frac{1}{(\mu-\lambda)^2}e^{-(\mu-\lambda)w}\right]_0^{\infty} = \lambda\left(\frac{\mu-\lambda}{\mu}\right)\frac{1}{(\mu-\lambda)^2}$$

$$\therefore \quad W_q = \frac{\lambda}{\mu(\mu-\lambda)} \quad ...(23.65)$$

(*iv*) To find expected waiting time in the system (including service time), W_s :

Since expected waiting time in the system = Expected waiting time in queue + expected service time, i.e. $W_s = W_q + 1/\mu$ (expected service time or mean service time = 1/μ). Substituting the value of W_q from eqn. (23.65), we get

$$W_s = \frac{\lambda}{\mu(\mu-\lambda)} + \frac{1}{\mu} = \frac{1}{\mu-\lambda} \quad ...(23.66)$$

(*v*) To find expected waiting time of a customer who has to wait, (*W* I *W* > 0).

The expected length of the busy period is given by

$$(W \mid W > 0) = \int_0^\infty w\Psi(w > 0)\, dw = \int_0^\infty w \,.\, (\mu - \lambda)\, e^{-(\mu - \lambda)w}\, dw \quad \text{[from eqn. (23.62)]}$$

Integrating by parts, we get

$$(\boldsymbol{W} \mid \boldsymbol{W} > \boldsymbol{0}) = \frac{\boldsymbol{1}}{\mu - \lambda} = \frac{\boldsymbol{1}}{\mu\,(\boldsymbol{1} - \rho)} \qquad ...(23.67)$$

(*vi*) **To find expected length of non-empty queue, ($L \mid L > 0$).**

By definition of conditional probability,

$(L \mid L > 0) = L_s/$ Prob. (an arrival has to wait, $L > 0$)

$= L_s/(1 - P_0)$ (since probability of an arrival not to wait is P_0)

Substituting the value of L_s and P_0 from eqns. (5.63) and (23.56), we get

$$(\boldsymbol{L} \mid \boldsymbol{L} > \boldsymbol{0}) = \frac{(\lambda/\mu)/(1 - \lambda/\mu)}{\lambda/\mu} = \frac{\mu}{\mu - \lambda} = \frac{\boldsymbol{1}}{\boldsymbol{1} - \rho} \qquad ...(23.68)$$

(*vii*) **To find the variance of queue length.** **[Kanpur 93]**

By definition,

$$\text{Var.}\{n\} = \sum_{n=1}^{\infty} n^2 P_n - \left(\sum_{n=1}^{\infty} n P_n\right)^2 = \sum_{n=1}^{\infty} n^2 P_n - [L_s]^2$$

$$= \sum_{n=1}^{\infty} n^2 (1 - \rho)\, \rho^n - \left(\frac{\rho}{1 - \rho}\right)^2 \qquad \text{(using (23.57) and (23.63))}$$

$$= (1 - \rho)\,[1^2\rho + 2^2\rho^2 + 3^2\rho^3 + ...] - \rho^2/(1 - \rho)^2$$

$$= (1 - \rho)\,\rho\,[1 + 2^2\rho + 3^2\rho^2 + ...] - \rho^2/(1 - \rho)^2$$

$$= (1 - \rho)\,\rho \left[\frac{1 + \rho}{(1 - \rho)^3}\right]^* - \frac{\rho^2}{(1 - \rho)^2} \qquad \textbf{[see foot note]}$$

$$= \rho/(1 - \rho)^2.$$

Q. 1. Obtain the steady state equations for the model ($M \mid M \mid 1$) : ($\infty \mid FCFS$) *i.e.* single server, Poisson arrival, negative exponential service), and also find the formula for :

(*i*) variance of the queue length, (*ii*) the average waiting length, (*iii*) Prob. Queue size $\geq N$,

(*iv*) The average (mean) queue length., (*v*) the average waiting length given that it is greater than zero,

(*vi*) The average number of customers in the system. **[Raj. Univ. (M. Phil) 90]**

2. Define the concept of busy period in queueing theory and obtain its distribution for the system $M \mid M \mid 1$: ($\infty \mid FCFS$).

Show that the average length of the busy period is $1/(\mu - \lambda)$.

3. Customers arrive at a sales counter in a Poisson fashion with mean arrival rate λ and exponential service times with mean service rate μ. Determine :

(*i*) Average length of non-empty queues, (*ii*) Average waiting time of an arrival.

4. For the queueing system in which there is a single channel and the inter-arrival time of units and the service time of units follow exponential distribution prove that :

(*i*) $1/(\mu - \lambda)$ is the average time an arrival spends in the system, (*ii*) $\lambda^2/\mu\,(\mu - \lambda)$ is average queue length.

(*viii*) **To find the probability of arrivals during the service time of any given customer.**

Since the arrivals are Poisson and service times are exponential, the probability of r arrivals during the service time of any given customer is given by

---

* Let $S = 1 + 2^2\rho + 3^2\rho^2 + ...$

Integrating both sides in the limit 0 to ρ

$$\int_0^\rho S\, d\rho = \rho + 2\rho^2 + ... = \rho(1 - \rho)^{-2}$$

[see foot-note on p. 832]

Now, differentiating w.r.t. 'ρ'

$$S = \frac{1}{(1 - \rho)^2} + \frac{2\rho}{(1 - \rho)^3} = \frac{(1 + \rho)}{(1 - \rho)^3}.$$

$$K_r = \int_0^\infty P_r(t)\, s(t)\, dt = \int_0^\infty \frac{e^{-\lambda t}(\lambda t)^r}{r!} \cdot \mu e^{-\mu t}\, dt = \frac{\lambda^r \mu}{r!} \int_0^\infty e^{-(\lambda+\mu)t}\, t^r\, dt$$

$$= \frac{\lambda^r \mu\, \Gamma(r+1)}{r!\, (\lambda+\mu)^{r+1}} \quad \left[\text{using } \int_0^\infty e^{-at} t^n\, dt = \frac{\Gamma(n+1)}{a^{n+1}}\right]$$

$$= \left(\frac{\lambda}{\lambda+\mu}\right)^r \cdot \frac{\mu}{\lambda+\mu} \qquad [\text{since } \Gamma(r+1) = r!]$$

23.12-1. Inter-Relationship Between L_s, L_q, W_s, W_q.

It can be proved under (rather) general conditions of arrival, departure, and service discipline that the formulae,

$$L_s = \lambda W_s, \qquad ...(23.69)$$

and

$$L_q = \lambda W_q, \qquad ...(23.70)$$

will hold in general. These formulae act as key points in establishing the strong relationships between W_s, W_q, L_s and L_q which can be found as follows.

By definition, $W_q = W_s - 1/\mu$. ...(23.71)

Thus, multiplying both sides by λ and substituting the values from (23.69) and (23.70),

$$L_q = L_s - \lambda/\mu. \qquad ...(23.72)$$

This means that one of the four expected values (together with λ and μ) should immediately yield the remaining three values.

Q. In (M | M | 1) : (∞ | FCFS) model obtain p.d.f. of waiting time (excluding service time) and hence obtain $E(W_q)$, $E(W_s)$, $E(L_q)$, $E(L_s)$. **[Garhwal M.Sc. (Stat.) 93]**

23.12-2. Illustrative Examples on Model I

Example 1. *A TV repairman finds that the time spent on his jobs has an exponential distribution with mean 30 minutes. If he repairs sets in the order in which they come in, and if the arrival of sets is approximately Poisson with an average rate of 10 per 8-hour day, what is repairman's expected idle time each day ? How many jobs are ahead of the average set just brought in ?*

[JNTU 2002; Agra 98, 93; Karnataka B.E. (CSE) 93; Meerut (Maths.) 91]

Solution. Here, $\mu = 1/30$, $\lambda = 10/(8 \times 60) = 1/48$. Therefore, expected number of jobs are

$$L_s = \frac{\lambda/\mu}{1-\lambda/\mu} = \frac{\lambda}{\mu-\lambda} = \frac{1/48}{1/30 - 1/48} = 1\tfrac{2}{3} \text{ jobs.}$$ **Ans.**

Since the fraction of the time the repairman is busy (*i.e.* traffic intensity) is equal to λ/μ, the number of hours for which the repairman remains busy in a 8-hour day is

$$= 8 \,.\, (\lambda/\mu) = 8 \times 30/48 = 5 \text{ hours.}$$

Therefore, the time for which the repairman remains idle in 8-hour day = (8 – 5) hours = 3 hours. **Ans.**

Example 2. *At what average rate must a clerk at a supermarket work in order to ensure a probability of 0.90 that the customer will not have to wait longer than 12 minutes ? It is assumed that there is only one counter to which customers arrive in a Poisson fashion at an average rate of 15 per hour. The length of service by the clerk has an exponential distribution.* **[Meerut (Maths.) 99, 96]**

Solution. Here, $\lambda = 15/60 = 1/4$ customer/minute, μ = ? Prob. [waiting time ≥ 12] = 1 – 0.90 = 0.10.

Therefore, $$\int_{12}^\infty \lambda\left(1 - \frac{\lambda}{\mu}\right) e^{-(\mu-\lambda)w}\, dw = 0.10 \quad \text{or} \quad \lambda\left(1 - \frac{\lambda}{\mu}\right)\left[\frac{e^{-(\mu-\lambda)w}}{-(\mu-\lambda)}\right]_{12}^\infty = 0.10$$

or $$e^{(3-12\mu)} = 0.4\,\mu \quad \text{or} \quad 1/\mu = 2.48 \text{ minute per service.}$$ **Ans.**

Example 3. *Arrivals at a telephone booth are considered to be Poisson, with an average time of 10 minutes between one arrival and the next. The length of a phone call assumed to be distributed exponentially with mean 3 minutes. Then,* (*a*) *What is the probability that a person arriving at the booth will have to wait ?* **[Meerut 2002; I.A.S. (Main) 91]**

(*b*) *What is the average length of the queues that form from time to time ?* **[Meerut 2002; Kanpur 2000]**

(*c*) *The telephone department will install a second booth when convinced that an arrival would expect to have to wait at least three minutes for the phone. By how much must the flow of arrivals be increased in order to justify a second booth ?* **[Meerut 2002; I.A.S. (Maths) 91]**

Solution. Here, $\lambda = 1/10$ and $\mu = 1/3$

(*a*) Prob. $(W > 0) = 1 - P_0 = \lambda/\mu$ [from eqn. (23.60)] $= \frac{1}{10} \times \frac{3}{1} = \frac{3}{10} = 0.3$ **Ans.**

(*b*) $(L \mid L > 0) = \mu/(\mu - \lambda) = \frac{1}{3}/(\frac{1}{3} - \frac{1}{10}) = 1.43$ persons [from eqn. (23.68)] **Ans.**

(*c*) $W_q = \lambda/\mu(\mu - \lambda)$ [from eqn. (23.65)]

Since $W_q = 3$, $\mu = \frac{1}{3}$, $\lambda = \lambda'$ (say) for second booth, therefore

$$3 = \frac{\lambda'}{\frac{1}{3}(\frac{1}{3} - \lambda')}, \quad \text{giving } \lambda' = 0.16.$$

Hence, increase in the arrival rate $= 0.16 - 0.10 = 0.06$ arrivals per minute. **Ans.**

Example 4. *As in Example 3, a telephone booth with Poisson arrivals spaced 10 minutes apart on the average, and exponential call lengths averaging 3 minutes.*

(*a*) *What is the probability that an arrival will have to wait more than 10 minutes before the phone is free ?*

(*b*) *What is the probability that it will take him more than 10 minutes altogether to wait for phone and complete his call ?*

(*c*) *Estimate the fraction of a day that the phone will be in use.*

(*d*) *Find the average number of units in the system.*

Solution. Here $\lambda = 0.1$ arrival per minute, $\mu = 0.33$ service per minute.

(*a*) Prob. [waiting time ≥ 10] $= \int_{10}^{\infty} \Psi(w)\, dw = \int_{10}^{\infty} \left(1 - \frac{\lambda}{\mu}\right) \lambda e^{-(\mu - \lambda)w}\, dw$

$$= -\frac{\lambda}{\mu}\left[e^{-(\mu - \lambda)w}\right]_{10}^{\infty} = 0.3e^{-2.3} = 0.03 \text{ [see Exponential Tables]}$$ **Ans.**

(*b*) Prob. [waiting time in the system ≥ 10]

$$= \int_{10}^{\infty} (\mu - \lambda)\, e^{-(\mu - \lambda)w}\, dw = e^{-10(\mu - \lambda)} = e^{-2.3} = 0.10 \text{ [see Exp. Table]}$$ **Ans.**

(*c*) The fraction of a day that the phone will be busy = traffic intensity $\rho = \lambda/\mu = 0.3$.

(*d*) Average number of units in the system,

$$L_s = \frac{\lambda}{\mu - \lambda} = \frac{\frac{1}{10}}{\frac{1}{3} - \frac{1}{10}} = \frac{3}{7} = 0.43 \text{ customer.}$$ **Ans.**

Example 5. (a) *In a railway marshalling yard, goods trains arrive at a rate of 30 trains per day. Assuming that the inter-arrival time follows an exponential distribution and the service time (the time taken to hump a train) distribution is also exponential with an average 36 minutes. Calculate the following :*

(*i*) *The average number of trains in the queue.*

(*ii*) *The probability that the queue size exceeds 10.* **[JNTU 99, 98; Raj. Univ. (M. Phil.) 93, 90]**

If the input of trains increases to an average 33 per day, what will be change in (i) and (ii) ?

Establish the formula you use in your calculations. **[JNTU 2002; Agra 98; I.A.S. (Main) 90]**

Solution. Here

$$\lambda = \frac{30}{60 \times 24} = \frac{1}{48} \text{ trains/minute; } \mu = \frac{1}{36} \text{ trains/minute, and } \rho = \frac{\lambda}{\mu} = \frac{36}{48} = 0.75.$$

(*i*) $$L_s = \frac{\rho}{1 - \rho} = \frac{0.75}{1 - 0.75} = 3 \text{ trains}.$$

(*ii*) Prob [*queue size* ≥ 10] $= \rho^{10} = (0.75)^{10} = 0.06.$ **Ans.**

When the input increases to 33 trains per day $\lambda = \frac{1}{43}$, $\mu = \frac{1}{36}$.

Therefore, $\rho = \lambda/\mu = 36/43 = 0.84$. Hence,

(*i*) $L_s = 0.84/0.16 = 5$ trains (*ii*) Prob (queue size ≥ 10) $= (0.84)^{10} = 0.2$ (approx.)

(b) *Trains arrive at the yard every 20 minutes and the service time is 40 minutes. If the line capacity of the yard is limited to 6, find*

(i) the probability the yard is empty.

(ii) the average number of trains in the system. **[JNTU (B. Tech.) 2003]**

Solution. Proceed as in part (a).

Example 6. *In the above problem calculate the following :*

(*i*) *Expected waiting time in the queue.*

(*ii*) *The probability that number of trains in the system exceeds 10.* **[JNTU (B. Tech.) 98]**

(*iii*) Average number of trains in the queue.

Solution. Here $\lambda = 1/48$, $\mu = 1/36$ and $\rho = 0.75$.

(*i*) Expected waiting time in the queue is

$$W_q = \frac{\lambda}{\mu(\mu - \lambda)} = \frac{1/48}{1/36\,(1/36 - 1/48)} = 108 \text{ minutes } \textit{or 1 hr 48 mts.}$$

(*ii*) $P(\geq 10) = \rho^{10} = (0.75)^{10} = 0.06$.

(*iii*) $L_q = \dfrac{\lambda^2}{\mu(\mu - \lambda)} = \dfrac{(1/48)^2}{1/36\,(1/36 - 1/48)} = \dfrac{108}{48} = 2.25$ or nearly 2 trains.

Example 7. *Consider an example from a maintenance shop. The inter-arrival times at toolcrib are exponential with an average time of 10 minutes. The length of the service time (amount of time taken by the toolcrib operator to meet the needs of the maintenance man) is assumed to be exponentially distributed, with mean 6 minutes. Find :*

(*i*) *The probability that a person arriving at the booth will have to wait.*

(*ii*) *Average length for the queue that forms and the average time that an operator spends in the Q-system.*

(*iii*) *The manager of the shop will install a second booth when an arrival would have to wait 10 minutes or more for the service. By how much must the rate of arrival be increased in order to justify a second booth.*

(*iv*) *The probability that an arrival will have to wait for more than 12 minutes for service and to obtain his tools.*

(*v*) *Estimate the fraction of the day that toolcrib operator will be idle.*

(*vi*) *The probability that there will be six or more operators waiting for the service.*

Solution. Here $\lambda = 60/10 = 6$ per hour, $\mu = 60/6 = 10$ per hour **[Virbhadrah 2000]**

(*i*) *A person will have to wait if the service facility is not idle.*

Probability that the service facility is idle = Probability of no customer in the system (P_0)

$\therefore$ Probability of waiting $= 1 - P_0 = 1 - (1 - \rho) = \rho = \lambda/\mu = 6/10 = 0.6$ **Ans.**

(*ii*) $L_q = \rho^2/(1 - \rho) = (0.6)^2/(1 - 0.6) = 0.9$

$L_s = L_q + \lambda/\mu = 0.9 + 0.6 = 1.5.$ $\therefore$ $W_s = L_s/\lambda = 1.5/6 = 1/4$ hours

(*iii*) $W_q = L_q/\mu = 0.9/6$ hrs. = 9 minutes.

Let λ be the arrival rate when a second booth is justified, *i.e.*, $W_q \geq 10$ minutes.

$$W_q = \frac{\lambda}{\mu(\mu - \lambda)} = \frac{10}{60}.$$

$\therefore$ $6\lambda = 10(10 - \lambda)$ or $16\lambda = 100$ or $\lambda = 6.25$.

Hence if the arrival rate exceeds 6.25 per hour, the second booth will be justified.

(*iv*) Probability of waiting for 12 minutes or more is given by

$$\text{Prob. } (W \geq 12) = \int_{12}^{\infty} \rho(\mu - \lambda)\, e^{-(\mu - \lambda)w} dw = -\rho\Big[e^{-(\mu - \lambda)w}\Big]_{12/60}^{\infty}$$

$$= \rho e^{-(\mu - \lambda)12/60} = 0.6\, e^{-(10 - 6).12/60} = 0.6\, e^{-4/5} = 0.27.$$ **Ans.**

(*v*) $P_0 = 1 - \rho = 0.4$, 40% of the time of toolcrib operator is idle.

(*vi*) Probability of six or more operators waiting for the service $= \rho^6 = (0.6)^6$.

Example 8. *On an average 96 patients per 24-hour day require the service of an emergency clinic. Also on average, a patient requires 10 minutes of active attention. Assume that the facility can handle only one emergency at a time. Suppose that it costs the clinic Rs. 100 per patient treated to obtain an average servicing time of 10 minutes, and that each minute of decrease in this average time would cost Rs. 10 per patient treated. How much would have to be budgeted by the clinic to decrease the average size of the queue from* $1\frac{1}{3}$ *patients to* $\frac{1}{2}$ *patient.* **[JNTU (B. Tech.) 2004; I.A.S. (Main) 93]**

Solution. Here $\lambda = \frac{96}{24 \times 60} = \frac{1}{15}$ patient/minute, $\mu = \frac{1}{10}$ patient/minute

Expected number of patients in the waiting line

$$L_q = \frac{\lambda^2}{\mu(\mu - \lambda)} = \frac{(1/15)^2}{1/10\,(1/10 - 1/15)} = 1\tfrac{1}{3} \text{ patients.}$$

But, $L_q = 1\frac{1}{3}$ is reduced to $L_q' = \frac{1}{2}$.

Therefore, substituting $L_q' = \frac{1}{2}$, $\lambda' = \lambda = 1/15$ in the formula $L_q' = \frac{\lambda'^2}{\mu'(\mu' - \lambda')}$, we get

$$\tfrac{1}{2} = \frac{(1/15)^2}{\mu'(\mu' - 1/15)}$$

which gives $\mu' = 2/15$ patient/minute.

Hence the average rate of treatment required is $1/\mu' = 7.5$ minutes.

Consequently, the decrease in the average rate of treatment $= 10 - 15/2 = 5/2$ minutes;
and the budget per patient $= 100 + 5/2 \times 10 =$ Rs. 125. So in order to get the required size of the queue, the budget should be increased from Rs. 100 to Rs. 125 per patient.

Example 9. *The mean rate of arrival of planes at an airport during the peak period is 20 hour, but the actual number of arrivals in any hour follows a Poisson distribution with the respective averages. When there is congestion, the planes are forced to fly over the field in the stack awaiting the landing of other planes that arrived earlier.*

(*i*) *How many planes would be flying over the field in the stack on an average in good weather and in bad weather ?*

(*ii*) *How long a plane would be in the stack and in the process of landing in good and in bad weather ?*

(*iii*) *How much stack and landing time to allow so that priority to land out of order would have to be requested only one time in twenty ?* **[Agra 98]**

Solution. Here

$$\lambda = 20 \text{ planes/hour}$$

and

$$\mu = \begin{cases} 60 \text{ planes/hour in good weather} \\ 30 \text{ planes/hour in bad weather} \end{cases}$$

(*i*) $L_q = \dfrac{\lambda^2}{\mu(\mu - \lambda)} = \begin{cases} 20^2/60(60-20) = 1/6 \text{ in good weather} \\ 20^2/30(30-20) = 1/3 \text{ in bad weather} \end{cases}$ **Ans.**

(*ii*) $W_s = \dfrac{1}{\mu - \lambda} = \begin{cases} 1/(60-20) = 1/40 \text{ hrs. in good weather} \\ 1/(30-20) = 1/10 \text{ hrs. in bad weather} \end{cases}$ **Ans.**

(*iii*) The waiting time 'w' taken by the plane in landing and in the stack is given by

$$\int_0^w (\mu - \lambda)\, e^{-(\mu - \lambda)w} dw = 0.95$$

or

$$(\mu - \lambda) \left[\frac{e^{-(\mu - \lambda)w}}{\lambda - \mu} \right]_0^w = 0.95$$

or

$$1 - e^{-(\mu - \lambda)w} = 0.95$$

or

$$e^{-(\mu - \lambda)w} = 0.05 \qquad \text{(A)}$$

Now the value of w can be determined in both the cases.

Case I. *When weather is good* :

Substituting $\lambda = 20$, $\mu = 60$ in eqn. (A), $e^{-40w} = 0.05$.

Taking the logarithm of both sides to the base e (instead of 10),

$$-40w = \log_e (0.05) = -2.9957 \qquad [\because \log_e e = 1, \therefore \log_e (0.05) = -2.9957]$$

or $$w = \frac{2.9957}{40} = 0.075 \text{ hour} = 4.5 \text{ minutes } \textbf{Ans.}$$

Case 2. *When weather is bad :*

Substitution $\lambda = 20$, $\mu = 30$ in eqn. (A),

$$e^{-10w} = 0.05$$

Solving this equation as in case I above, we get

$$w = 0.3 \text{ hour} = 18 \text{ minutes.}$$ **Ans.**

Example 10. *A refinery distributes its products by trucks, loaded at the loading dock. Both company-trucks and independent distributor's trucks are loaded. The independent firms complained that sometimes they must wait in line and thus lose money paying for a truck and driver, that is only waiting. They have asked the refinery either to put in a second loading dock or to discount prices equivalent to the waiting time. Extra loading dock cost Rs. 100/- per day whereas the waiting time for the independent firms cost Rs. 25/- per hour. The following data have been accumulated. Average arrival rate of all trucks is 2 per hour and average service rate is 3 per hour. Thirty per cent of all trucks are independent. Assuming that these rates are random according to the Poisson distrbutions, determine :*

(*a*) *the probability that a truck has to wait*

(*b*) *the waiting time of a truck that waits, and*

(*c*) *the expected cost of waiting time of independent trucks per day.*

Is it advantageous to decide in favour of a second loading dock to ward off the complaints ?

Solution. We are given that $\lambda = 2$ per hour and $\mu = 3$ per hour.

(*a*) The probability that a truck has to wait for scrvice is the utilization factor,

$$\rho = \frac{\lambda}{\mu} = \frac{2}{3} = 0.66.$$

(*b*) The waiting time of a truck that waits is

$$(W \mid W > 0) = \frac{W_s}{\text{Prob } (W > 0)} = \left[\frac{\lambda}{\mu(\mu - \lambda)} / (\lambda/\mu)\right]$$

$$= \frac{1}{\mu - \lambda} = \frac{1}{3 - 2} = 1 \text{ hour.}$$

(*c*) The total expected waiting time of independent trucks per day is given by :

Expected waiting time = Trucks per day × % Independent truck × Expected waiting time per truck

$$= (2 \times 8)\ (0.3W_q) = 16 \times 0.3 \times \frac{\lambda}{\mu(\mu - \lambda)} = 4.8 \times \frac{2}{3(3 - 2)} = 3.2 \text{ hour per day.}$$

$\therefore$ Expected Cost = Rs. (3.2 × 25) = Rs. 80.

Example 11. (a) *Barber A takes 15 minutes to complete one hair cut. Customers arrive in his shop at an average rate of one every 30 minutes and the arrival process is Poisson. Barber B takes 25 minutes to complete one hair-cut and customers arrive in his shop at an average rate of one every 50 minutes, the arrival process being Poisson during steady state.*

(*i*) *Where would you expect the bigger queue ?*

(*ii*) *Where would you require more times waiting included, to complete a hair-cut.*

(b) *In a hair dressing salon with one barber the customer arrival follows Poission distribution at an average rate of one every 45 minutes. The service time is exponentially distributed with a mean of 30 minutes. Find (i) Average number of customers in the salon.*

(ii) Average waiting time of a customer before service.

(iii) Average idle time of the barber. **[VTU (BE Mech.) 2003]**

Solution. Proceed as in above solved examples.

Example 12. (a) *An airlines organisation has one reservation clerk on duty in its local branch at any given time. The clerk handles information regarding passenger reservation and flight timings. Assume that the number of* *customers arriving during any given period is Poisson distributed with an arrival rate of eight per hour and that the reservation clerk can serve a customer in six minutes on an average, with an exponentially distributed service time.*

(i) What is the probability that the system is busy ?

(ii) What is the average time a customer spends in the system ?

(iii) What is the average length of the queue and what is the number of customers in the system ? **[C.A., Nov. 96]**

(b) *If for a period of 2 hours in a day (8–10 AM) planes arrive at the aerodrome for every 20 minutes but the service time continues to remain 32 minutes then calculate for this period :*

(i) the probability that the aerodrome is empty (ii) average queue length on the assumption that the time capacity of the aerodrome is limited to 6 planes. **[JNTU (Mech. & Prod.) May 2004]**

Solution. (a) According to the given information :

Mean arrival rate, $\lambda = 8$ customers per hour

Mean service rate, $\mu = \frac{60}{6} = 10$ customers per hour

$$\therefore \quad \rho = \frac{\lambda}{\mu} = \frac{8}{10} \quad \text{or} \quad \frac{4}{5}$$

(i) The probability that the system is busy is given by :

$$1 - P_0 = 1 - \left(1 - \frac{\lambda}{\mu}\right) = \frac{\lambda}{\mu} = 0{\cdot}8 \text{, } i.e., 80\% \text{ of the time system is busy.}$$

(ii) The average time a customer spends in the system is given by :

$$W_s = \frac{1}{\mu - \lambda} = \frac{1}{10 - 8} = \frac{1}{2} \text{ hours or 30 minutes}$$

(iii) The average length of the queue is given by :

$$L_q = \frac{\lambda}{\mu} \times \frac{\lambda}{\mu - \lambda} = \frac{8}{10} \times \frac{8}{10 - 8} = 3{\cdot}2 \text{ customers}$$

The average number of customers in the system is given by :

$$L_s = \frac{\lambda}{\mu - \lambda} = \frac{8}{10 - 8} \text{ or 4 customers.}$$

(b) [**Hint :** Proceed as (a)]

Example 13. *Customers arrive at a sales counter manned by a single person according to a Poisson process with a mean rate of 20 per hour. The time required to serve a customer has an exponential distribution with a mean of 100 seconds. Find the average waiting time of a customer.* **[Poona (M.B.A.) 98]**

Solution. Here we are given :

$$\lambda = 20 \text{ per hour}, \mu = \frac{60 \times 60}{100} = 36 \text{ per hour}$$

The average waiting time of a customer in the queue is given by :

$$W_q = \frac{\lambda}{\mu(\mu - \lambda)} = \frac{20}{36(36 - 20)} = \frac{5}{36 \times 4} \text{ hours or } \frac{5 \times 3600}{36 \times 4}, \ i.e., \text{ 125 seconds.}$$

The average waiting time of a customer in the system is given by :

$$W_s = \frac{1}{(\mu - \lambda)} = \frac{1}{(36 - 20)} \text{ or } \frac{1}{16} \text{ hour } i.e., \text{ 225 seconds.}$$

Example 14. *Customers arrive at a one-window drive according to a Poisson distribution with mean of 10 minutes and service time per customer is exponential with mean of 6 minutes. The space in front of the window can accommodate only three vehicles including the serviced one. Other vehicles have to wait outside this space. Calculate :*

(i) Probability that an arriving customer can drive directly to the space in front of the window.

(ii) Probability that an arriving customer will have to wait outside the directed space.

[JNTU (B. Tech.) 2003; SJMIT (BE Mech.) 2002; C.A. (May) 98]

(iii) How long an arriving customer is expected to wait before getting the service ?

Solution. From the given information, we find that :

Mean arrival rate, $\lambda = 6$ customers per hour

and mean service rate, $\mu = 10$ customers per hour

(i) Probability that an arriving customer can drive directly to the space in front of the window is given by :

$$P_0 + P_1 + P_2 = \left(1 - \frac{\lambda}{\mu}\right) + \frac{\lambda}{\mu}\left(1 - \frac{\lambda}{\mu}\right) + \left(\frac{\lambda}{\mu}\right)^2\left(1 - \frac{\lambda}{\mu}\right)$$

$$= \left(1 - \frac{\lambda}{\mu}\right)\left[1 + \frac{\lambda}{\mu} + \left(\frac{\lambda}{\mu}\right)^2\right]$$

$$= \left(1 - \frac{6}{10}\right)\left[1 + \frac{6}{10} + \left(\frac{6}{10}\right)^2\right] = \frac{98}{1225} \text{ or } 0{\cdot}784$$

(ii) Probability that an arriving customer will have to wait outside the directed space is given by :

$$1 - (P_0 + P_1 + P_2) = 1 - 0{\cdot}784 = 0{\cdot}216 \text{ or } 21{\cdot}6\%$$

(iii) Expected waiting time of a customer being getting the service is given by :

$$W_q = \frac{\lambda}{\mu(\mu - \lambda)} = \frac{6}{10(10-6)} = \frac{3}{20} \text{ hr. or 9 minutes.}$$

Example 15. *The rate of arrival of customers at a public telephone booth follows Poisson distribution, with an average time of 10 minutes between one customer and the next. The duration of a phone call is assumed to follow exponential distribution, with mean time of 3 minutes.*

(i) What is the probability that a person arriving at the booth will have to wait ?

(ii) What is the average length of the non-empty queues that form from time to time ?

(iii) The Mahanagar Telephone Nigam Ltd. will install a second booth when it is convinced that the customers would expect waiting for at least 3 minutes for their turn to make a call. By how much time should the flow of customers increase in order to justify a second booth ?

(iv) Estimate the fraction of a day that the phone will be in use. **[C.A., (May) 1999; Delhi (M. Com.) 99]**

Solution. Here we are given :

$$\lambda = \frac{1}{10} \times 60 \text{ or 6 per hour and } \mu = \frac{1}{3} \times 60 \text{ or 20 per hour}$$

(i) Probability that a person arriving at the booth will have to wait

$$= 1 - P_0 = 1 - \left(1 - \frac{\lambda}{\mu}\right) = \frac{6}{20} \text{ or } 0{\cdot}3.$$

(ii) Average length of non-empty queues

$$= \frac{\mu}{\mu - \lambda} = \frac{20}{20-6} = 1{\cdot}42.$$

(iii) The installation of a second booth will be justified if the arrival rate is greater than the waiting time. Now, if λ' denotes the increased arrival rate, expected waiting time :

$$W_q' = \frac{\lambda'}{\mu(\mu - \lambda')} \Rightarrow \frac{3}{60} = \frac{\lambda'}{20(20-\lambda')} \text{ or } \lambda' = 10.$$

(ii) P_0 = Prob. of no customer in the system $= 1 - \frac{\lambda}{\mu} = 0{\cdot}5$

Thus 50% of time an arrival will not have to wait.

(iii) Average time spent by a customer $= \frac{1}{\mu - \lambda} = \frac{1}{5}$ hour or 12 minutes

(iv) Average queue length $= \frac{\lambda^2}{\mu(\mu - \lambda)} = \frac{5 \times 5}{10(10-5)} = 0{\cdot}5$

(v) The management will deploy the person exclusively for Xeroxing when the average time spent by a customer exceeds 15 minutes. We wish to calculate the arrival rate that will lead to such situation. Let this arrival rate be λ'. Then

$$\frac{1}{\mu - \lambda'} > \frac{15}{60} \text{ or } \frac{1}{10 - \lambda'} > \frac{1}{4}, \text{ i.e., } \lambda' > 6.$$

Hence, if the arrival rate of customers is greater than 6 customers per hour, the average time spent by a customer will exceed 15 minutes.

Example 16. *Telephone users arrive at a booth following a Poisson distribution with an average time of 5 minutes between one arrival and the next. The time taken for a telephone call is on an average 3 minutes and it* *follows an exponential distribution. What is the probability that the booth is busy ? How many more booths should be established to reduce the waiting time to less than or equal to half of the present waiting time ?*

[JNTU (B. Tech.) 2003; Nagpur (M.B.A.) Nov. 98; Madras (M.B.A.), Dec. 97]

Solution. Here we are given :

arrival rate, λ = 12 per hour

service rate, μ = 20 per hour

Probability that booth is busy

$$= 1 - P_0 = \frac{\lambda}{\mu} = \frac{12}{20} = 0{\cdot}60$$

Average waiting time in queue :

$$W_q = \frac{\lambda}{\mu(\mu - \lambda)} = \frac{12}{20(20-12)} = \frac{3}{40} \text{ hour}$$

Average waiting time in system :

$$W_s = \frac{1}{\mu - \lambda} = \frac{1}{20-12} = \frac{1}{8} \text{ hour}$$

In case waiting time is required to be reduced to half,

$$W_s' = \frac{1}{\mu' - \lambda} \Rightarrow \frac{1}{16} = \frac{1}{\mu' - 12} \text{ or } \mu' = 28 \text{ per hour.}$$

Example 17. *The XYZ company's quality control deptt. is managed by a single clerk, who takes on an average 5 minutes in checking parts of each of the machine coming for inspection. The machines arrive once in every 8 minutes on the average. One hour of the machine is valued at Rs. 15 and a clerk's time is valued at Rs. 4 per hour. What are the average hourly queuing system costs associated with the quality control department ?*

[Poona (M.B.A.) 95]

Solution. Mean arrival rate, $\lambda = \frac{1}{4}$ per min. $= \frac{60}{8}$ per hour

Mean service rate, $\mu = \frac{1}{5}$ per min. = 12 per hour

Average time spent by a machine in the system

$$= \frac{1}{\mu - \lambda} = \frac{2}{9} \text{ hour}$$

Average queuing cost per machine is $\frac{15 \times 2}{9}$ = Rs. $\frac{10}{3}$

Average arrival of $\frac{60}{8}$ machines per hour costs $\frac{10}{3} \times \frac{60}{8}$ or Rs. 25

Average hourly queuing cost = Rs. 25.

Average hourly cost for the clerk = Rs. 4

Hence total cost = Rs. 29 per hour.

Example 18. *A company distributes its products by trucks loaded at its only loading station. Both, company's trucks and contractor's trucks, are used for this purpose. It was found out that on an average every five minutes, one truck arrived and the average loading time was three minutes. 50% of the trucks belong to the contractor. Find out :*

(i) the probability that a truck has to wait,

(ii) the waiting time of truck that waits, and

(iii) the expected waiting time of contractor's trucks per day, assuming a 24-hours shift.

[Punjabi (M.B.A.) 99; C.A. Nov. 96]

Solution. Here we are given :

Average arrival rate of trucks, $\lambda = \frac{60}{5}$ = 12 trucks/hr.

Average service rate of trucks, $\mu = \frac{60}{3}$ = 20 trucks/hr.

(i) The probability that a truck has to wait is given by :

$$\rho = \frac{\lambda}{\mu} = \frac{12}{20} = 0{\cdot}6$$

(ii) The waiting time of a truck that waits is given by :

$$W_s = \frac{1}{\mu - \lambda} = \frac{1}{20 - 12} = \frac{1}{8} \text{ hour or } 7{\cdot}5 \text{ minutes.}$$

(iii) The expected waiting time of contractor's truck per day (assuming 24 hrs. shift)

= (No. of trucks per day) × (Contractor's percentage) × (Expected waiting time of a truck)

$$= 12 \times 24 \times \frac{50}{100} \times \frac{\lambda}{\mu(\mu - \lambda)}$$

$$= 288 \times \frac{1}{2} \times \frac{12}{20 \times 8} = \frac{54}{5} \text{ or } 10{\cdot}8 \text{ hrs.}$$

Example 19. *A warehouse has only one loading dock manned by a three person crew. Trucks arrive at the loading dock at an average rate of 4 trucks per hour and the arrival rate is Poisson distributed. The loading of a truck takes 10 minutes on an average and can be assumed to be exponentially distributed. The operating cost of a truck is Rs. 20 per hour and the members of the loading crew are paid @ Rs. 6 each per hour. Would you advise the truck owner to add another crew of three persons ?* **[Delhi (M. Com.) 96]**

Solution. Total hourly cost = Loading crew cost + cost of waiting time

With present crew :

Loading cost = No. of loaders × Hourly wage rate = Rs. 18/hour.

Waiting time cost

$$= \left[\begin{array}{c}\textit{Expected waiting} \\ \textit{time per truck } (W_s)\end{array}\right] \times \left[\begin{array}{c}\textit{Expected arrivals} \\ \textit{per hour } (\lambda)\end{array}\right] \times \left[\begin{array}{c}\textit{Hourly} \\ \textit{waiting cost}\end{array}\right]$$

$$= \frac{4}{6 - 4} \times 20 = \text{Rs. } 40/\text{hr.}$$

Total cost = Rs. 18 + Rs. 40 = Rs. 58/hr.

After proposed crew addition :

$$\text{Total cost} = 6 \times 6 + \frac{4}{(12 - 4)} \times 20 = \text{Rs. } 46/\text{hr.}$$

Hence it will be beneficial to add a crew of 3 loaders.

Example 20. *A road transport company has one reservation clerk on duty at a time. She handles information of bus schedules and makes reservations. Customers arrive at a rate of 8 per hour and the clerk can service 12 customers on an average per hour. After stating your assumptions, answer the following :*

(i) What is the average number of customers waiting for the service of the clerk ?

(ii) What is the average time a customer has to wait before getting service ?

(iii) The management is contemplating to install a computer system to handle the information and reservations. This is expected to reduce the service time from 5 to 3 minutes. The additional cost of having the new system works out to be 12 paise per minute spent waiting before being served, should the company install the computer system ? Assume 8-hour working day. **[Madras (M.B.A.) 97]**

Solution. (i) Here we are given :

customer arrival rate = λ = 8 per hour and

service rate = μ = 12 per hour.

Average number of customers waiting for the service of the clerk (in the system) :

$$L_s = \frac{\lambda}{\mu - \lambda} = \frac{8}{12 - 8} = 2 \text{ customers.}$$

The average number of customers waiting for the service of the clerk (in the queue) :

$$L_q = \frac{\lambda^2}{\mu(\mu - \lambda)} = \frac{8 \times 8}{12(12 - 8)} \text{ or } 1{\cdot}33 \text{ customers.}$$

(ii) The average waiting time of a customer (in the system) before getting service :

$$W_s = \frac{1}{\mu - \lambda} = \frac{1}{12 - 8} \text{ hour or } 15 \text{ minutes.}$$

(iii) The average waiting time of a customer (in the queue) before getting service :

$$W_s = \frac{\lambda}{\mu(\mu-\lambda)} = \frac{8}{12(12-8)} = \frac{1}{6} \text{ hour or 10 minutes.}$$

(iv) Here calculate the difference between the goodwill cost of customers with one system and the goodwill cost of customers with an additional computer system. This difference will be compared with the additional cost (of Rs. 50 per day) of installing another computer system.

Now an arrival waits for W_q hours before being served and there are λ arrivals per hour. Thus expected waiting time for all customers in an 8 hour day with one system $= 8\lambda.\ W_q = 8 \times 8 \times \frac{1}{6}$ hrs. or $\frac{64}{6} \times 60$ minutes, *i.e.*, 640 minutes.

The goodwill cost per day with one system

$$= \text{Rs. } 640 \times \frac{12}{100} = \text{Rs. } 76{\cdot}80$$

The expected waiting time of a customer before getting service when there is an additional computer system is :

$$W_q' = \frac{8}{20(20-8)} = \frac{8}{20 \times 12} \text{ or } \frac{1}{30} \text{ hr.}$$

Thus expected waiting time of customers in an 8-hour day with an additional computer system is $8\lambda.\ W_q'$

$$= 8 \times 8 \times \frac{1}{30} \text{ hrs.} = 128 \text{ minutes.}$$

The total goodwill cost with an additional computer system

$$= \text{Rs. } 128 \times \frac{12}{100} = \text{Rs. } 15{\cdot}36.$$

Hence reduction in goodwill cost with the installation of a computer system

$$= \text{Rs. } 76{\cdot}80 - \text{Rs. } 15{\cdot}36 = \text{Rs. } 61{\cdot}44.$$

Whereas the additional cost of a computer system is Rs. 50 per day, Rs. 61·44 is the reduction in goodwill cost when additional computer system is installed, hence there will be net saving of Rs. 11·44 per day. It is, therefore, worthwhile to instal a computer.

Example 21. *In the production shop of a company, the breakdown of the machines is found to be Poisson distributed with an average rate of 3 machines per hour. Breakdown time at one machine costs Rs. 40 per hour to the company. There are two choices before the company for hiring the repairmen. One of the repairmen is slow but cheap the other fast but expensive. The slow-cheap repairman demands Rs. 20 per hour and will repair the broken down machines exponentially at the rate of 4 per hour. The fast-expensive repairman demands Rs. 30 per hour and will repair machines exponentially at an average rate of 6 per hour, which repairman should be hired ?* **[AIMS (BE Ind.) Bangl. 2002; Gujarat (M.B.A.) 97]**

Solution. Here we compare the total expected hourly cost for both the repairmen which would equal the total wages paid plus the cost due to machine breakdown (*i.e.*, for the non-productive machine hours).

Cost of non-productive time

$$= \text{Average number of machines in the system} \times \text{Cost of idle machine hour}$$

$$= L_s \times (\text{Rs. } 40/\text{hour}) = \frac{\lambda}{\mu - \lambda} \times 40.$$

For *slow-cheap repairman* :

$$\lambda = 3 \text{ machines per hour}, \quad \mu = 4 \text{ machines per hour}$$

$$\therefore \quad L_s = \frac{3}{4-3} = 3 \text{ machines.}$$

Cost of non-productive machine time $= 40 \times 3 =$ Rs. 120.

$\therefore$ Total cost of slow but cheap repairman $=$ Rs. $40 \times 3 +$ Rs. $20 =$ Rs. 140.

For *fast-expensive repairman* :

$$\lambda = 3 \text{ machines per hour}$$

$\mu = 6$ machines per hour

$L_s = \dfrac{3}{6-3} = 1$ machine.

Thus, the cost of non-productive machine time

$= 40 \times 1 = $ Rs. 40.

$\therefore$ Total cost of fast but expensive repairman

$=$ Rs. 40×1 + Rs. 30 = Rs. 70.

Obviously, the *fast repairman* should be employed by the company.

Example 22. *In a factory, the machine breakdown on an average rate is 10 machines per hour. The idle time cost of a machine is estimated to be Rs. 20 per hour. The factory works 8 hours a day. The factory manager is considering 2 mechanics for repairing the machines. The first mechanic A takes about 5 minutes on an average to repair a machine and demands wages Rs. 10 per hour. The second mechanic B takes about 4 minutes in repairing a machine and demands wages at the rate of Rs. 15 per hour. Assuming that the rate of machine breakdown is Poisson-distributed and the repair rate is exponentially distributed, which of the two mechanics should be engaged?* **[Delhi (M. Com.) 97]**

Solution. Here we shall compare the expected daily cost *viz.*, total wages paid plus cost due to machine breakdown for both the repairman.

Total wages for fast repairman = Hourly rate $\times$ No. of hours

$= 10 \times 8 = $ Rs. 80

Total wages for slow repairman = $15 \times 8 = $ Rs. 120.

Cost of non-productive time

= (Average number of machines in the system)

$\times$ (Cost of idle machine hour) $\times$ (No. of hours)

$$= \frac{\lambda}{\mu - \lambda} \times 20 \times 8.$$

For *fast repairman* :

$\lambda = 10$ machines per hour, $\mu = 12$ machines per hour.

$$\therefore \quad \text{Total cost} = 80 + \frac{10}{(12-10)} \times 20 \times 8 = \text{Rs. } 880.$$

For *slow repairman* :

$\lambda = 10$ machines per hour, $\mu = 12$ machines per hour.

$$\therefore \quad \text{Total cost} = 120 + \frac{10}{(15-10)} \times 20 \times 8 = \text{Rs. } 440.$$

Obviously, the fast repairman should be employed by the company.

Example 23. *A firm has several machines and wants to instal its own service facility for the repair of its machines. The average breakdown rate of the machines is 3 per day. The repair time has exponential distribution. The loss incurred due to the lost time of an inoperative machine is Rs. 40 per day. There are two repair facilities available. Facility X has an installation cost of Rs. 20,000 and facility Y costs Rs. 40,000. The total labour cost per year for the two facilities is Rs. 5,000 and Rs. 8,000 respectively. Facility X can repair 4 machines daily while facility Y can repair 5 machines daily. The life span of both the facilities is 4 years. Which facility shoud be installed?* **[Kurukshetra (M.B.A.) Nov. 96]**

Solution. We shall compare the total annual cost of the two facilities by using the relation :

Total annual cost $= \dfrac{1}{4}$ (cost investment expenditure)

+ (annual labour cost) + (annual cost of lost revenue due to down machines).

Facility X : Annual capital cost $= \dfrac{1}{4}(20{,}000) = $ Rs. 5,000

Annual labour cost = Rs. 5,000

Expected number of customers in the system

$$= \frac{\lambda}{\mu - \lambda} = \frac{3}{4-3} = 3 \text{ per day.}$$

The daily cost of lost time $= 3 \times 40 = $ Rs. 120 per day

Annual cost of lost machine time $= 365 \times 120 = $ Rs. 43,800.

Total annual cost for facility X $= 5{,}000 + 5{,}000 + 43{,}800 = $ Rs. 53,800.

Facility Y: Annual capital cost $= \frac{1}{4}(40{,}000) = $ Rs. 10,000

Annual labour cost = Rs. 8,000.

Expected number of customers in the system $= \frac{3}{5-3} = \frac{3}{2}$ per day.

The daily cost of lost time = Rs. $40 \times \frac{3}{2}$ = Rs. 60.

Annual cost of lost machine-time $= 365 \times 60 = $ Rs. 21,900.

Total annual cost for facility Y = Rs. 39,900.

Hence, facility Y should be preferred.

Example 24. *A tax consulting firm has four service stations (counters) in its office to receive people who have problems and complaints about their income, wealth and sales taxes. Arrivals average 100 persons in a 10-hour service day. Each tax adviser spends an irregular amount of time servicing the arrivals which have been found to have an exponential distribution. The average service time is 20 minutes. Calculate :*

(i) the average number of customers in the system,

(ii) average number of customers waiting to be serviced,

(iii) average time a customer spends in the system,

(iv) average waiting time for a customer,

(v) the probability that a customer has to wait before he gets service. **[Virbhadrah 2000; Punjab (M.B.A.) 98]**

Solution. Here we are given :

$$\lambda = 10/\text{hour}, \mu = 3/\text{hour}, k = 4 \text{ and } \rho = \frac{\lambda}{\mu} = \frac{10}{12}$$

Probability of no customer in the system is :

$$P_0 = \left[\sum_{n=0}^{k-1} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n + \frac{1}{k!}\left(\frac{\lambda}{\mu}\right)^k \frac{1}{\{1-(\lambda/k\mu)\}} \right]^{-1}$$

$$= \left[1 + \frac{10}{3} + \frac{1}{2}\left(\frac{10}{3}\right)^2 + \frac{1}{6}\left(\frac{10}{3}\right)^3 + \frac{1}{24}\left(\frac{10}{3}\right)^4 \frac{1}{\{1-(10/12)\}} \right]^{-1}$$

$$= 0{\cdot}0208$$

(i) Average number of customers in the system is :

$$L_s = L_q + \frac{\lambda}{\mu} = \left[\frac{1}{(k-1)!}\left(\frac{\lambda}{\mu}\right)^k \frac{\lambda\mu}{(k\mu-\lambda)^2} \right] P_0 + \frac{\lambda}{\mu}$$

$$= \left[\frac{1}{3!}\left(\frac{10}{3}\right)^4 \frac{30}{(12-10)^2} \right] \times 0{\cdot}0208 + \frac{10}{3} = 6{\cdot}567$$

(ii) Average queue length is given by :

$$L_q = L_s - \frac{\lambda}{\mu} = 6{\cdot}567 - \frac{10}{3} = 3{\cdot}234 \text{ customers}$$

(iii) Average time a customer spends in the system is :

$$W_s = W_q + \frac{1}{\mu} = \frac{L_q}{\lambda} + \frac{1}{\mu}$$

$$= \frac{3{\cdot}234}{10} + \frac{1}{3} = 0{\cdot}6567 \text{ hour}$$

(iv) Average time a customer waits for service in the queue is given by :

$$W_q = \frac{L_q}{\lambda} = \frac{3{\cdot}234}{10} = 0{\cdot}3234 \text{ hour}$$

(v) Probability that a customer has to wait is :

$$P(n \geq k) = \frac{1}{k!}\left(\frac{\lambda}{\mu}\right)^k \frac{1}{[1-(\lambda/k\mu)]} P_0$$

$$= \frac{1}{4!}\left(\frac{10}{3}\right)^4 \frac{1}{[1-(10/12)]}(0{\cdot}0208) = 0{\cdot}618.$$

Example 25. *A bank has two tellers working on savings accounts. The first teller handles withdrawals only. The second teller handles deposits only. It has been found that the service time distribution for both deposits and withdrawals is exponential with mean service time 3 minutes per customer. Depositors are found to arrive in Poisson fashion throughout the day with mean arrival rate of 16 per hour. Withdrawers also arrive in Poisson fashion with mean arrival rate of 14 per hour. What would be the effect on the average waiting time for depositors and withdrawers if each teller could handle both withdrawals and deposits ? What could be the effect if this could be accomplished by increasing the mean service time to* 3·5 *minutes* ? **[A.I.M.A. (P.G. Dip. in Management), Dec. 98]**

Solution. Initially, we have two independent queuing systems for withdrawers and depositors with input as Poisson distribution and service as exponential distribution.

For *withdrawers* : $\lambda = 14/\text{hour}$; $\mu = 3/\text{minute}$ or $20/\text{hour}$

Average waiting time in queue, $W_q = \dfrac{\lambda}{\mu(\mu-\lambda)}$

$$= \frac{14}{20(20-14)} = \frac{14}{20 \times 6} = \frac{7}{60} \text{ hour or 7 minutes.}$$

For depositors : $\lambda = 16/\text{hour}$; $\mu = 3/\text{minute}$ or $20/\text{hour}$.

Average waiting time in queue, $W_q' = \dfrac{\lambda}{\mu(\mu-\lambda)}$

$$= \frac{16}{20(20-16)} = \frac{16}{20 \times 4} = \frac{1}{5} \text{ hour or 12 minutes.}$$

If each teller could handle both withdrawals and deposits, we have a common queue with two servicers. The queuing system is thus with 2 service channels with $\lambda = 14 + 16 = 30/\text{hour}$ and $\mu = 20/\text{hour}$.

Average waiting time of arrival in the queue is :

$$W_q = \frac{L_q}{\lambda} = \frac{1}{(k-1)!}\left(\frac{\lambda}{\mu}\right)^k \frac{\mu}{(k\mu-\lambda)^2} P_0$$

where $$P_0 = \left[\sum_{n=0}^{k-1} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n + \frac{1}{k!}\left(\frac{\lambda}{\mu}\right)^k \frac{k\mu}{k\mu-\lambda}\right]^{-1}.$$

EXAMINATION PROBLEMS (ON MODEL I)

1. Customers arrive at a box office window, being manned by a single individual, according to a *Poisson* input process with a mean rate of 20 per hour. The time required to serve a customer has an exponential distribution with a mean of 90 seconds. Find the average waiting time of a customer.

   Also determine the average number of customers in the system and average queue length.

   [**Hint.** Here $\lambda = 0.5$, $\mu = 0.67$. Use formula for W_q, L_s. and L_q.]

   [**Ans.** $W_q = 4.5$ min/customer, $L_s = 3$, $L_q = 2.25$]

2. Arrivals of machinists at a toolcrib are considered to be *Poisson* distributed at an average rate of 6 per hour. The length of time the machinists must remain at the toolcrib is exponentially distributed with the average time being 0.05 hour.

   (a) What is the probability that a machinist arriving the toolcrib will have to wait ?

   (b) What is the average number of machinists at the toolcrib ?

   (c) The company will install a second toolcrib when conviced that a machinist would expect to have spend at least 6 minutes waiting and being serviced at the toolcrib. By how much must the flow of machinists to the toolcrib increase to justify the addition of a second toolerib ?

[**Hint.** Here $\lambda = \frac{6}{60}$, $\mu = \frac{1}{05 \times 60} = \frac{1}{3}$. To find W_s proceed as solved ***Example* 3**.]

[**Ans.** (a) $P(W > 0) = \frac{3}{10}$, (b) $(L \mid L > 0) = 1.43$, (c) $\lambda' - \lambda = \frac{2}{9} - \frac{1}{10} = \frac{11}{90}$.]

3. Consider a self service store with one cashier. Assume *Poisson* arrivals and exponential service times. Suppose that 9 customers arrive on the average every 5 minutes and the cashier can serve 10 in 5 minutes. Find :
(*i*) the average number of customers queueing for service.,
(*ii*) the probability of having more than 10 customers in the system,
(*iii*) the probability that a customer has to queue for more than 2 minutes.
If the service can be speeded up to 12 in 5 minutes by using a different cash register, what will be the effect on the quantities in (*i*), (*ii*) and (*iii*).
[**Hint.** Case I. $\lambda = 9/5$ per min., $\mu = 10/5$ per min. $L_s = 3$, $P(\geq 10) = (0.9)^{10}$, $P(W > 2) = 0.67$.
Case II. $\mu = 12/5$ (instead of $10/5$), $\mu = 9/5$.
$L_s = 3$, $P(\geq 10) = (0.75)^{10}$, $P(W > 2) = 0.30$.]
[**Ans.** Since the average number of customers is reduced to 3 and the probability that a customer has to wait for more than 2 minutes is also reduced to 0.30, the case II will be preferable].

4. In a bank there is only one window, a solitary employee performs all the service required and the window remains continuously open from 7.00 (a.m.) to 1.00 (p.m.). It has been discovered that the average number of clients is 54 during the day and that the average service time is of five minutes per person. Calculate :
(*i*) average number of clients in the system (including the one being served)
(*ii*) the average number of clients in the waiting line (excluding the one being served), and
(*iii*) the average waiting time.
[**Hint.** $\lambda = 54/6 \times 60$, $\mu = 11/5$. Use formulae for L_s, L_q, W_s.]
[**Ans.** $L_s = 3$, $L_q = 2.25$, $W_s = 20$ min. per client.]

5. A ticket issuing office is being manned by a single server. Customers arrive to purchase tickets according to a Poisson process with a mean rate of 30 per hour. The time required to serve a customer has an exponential distribution with a mean of 90 seconds. Find the value of P_n, L_s and W_s, where L_s and W_s denote the expected line length and waiting time in the system respectively:]
[**Hint.** $\lambda = 30/60 = 1/2$, $\mu = 60/90 = 2/3$. Use formulae for P_n, L_s, W_s]
[**Ans.** $P_n = 1/4\ (1/4)^n$, $L_s = 3$ customers, $W_s = 6$ min.]

6. An airline has one reservation clerk on duty at a time. He handles in formation about flight schedules and makes reservations. All calls to the airline are answered by an operator. If a caller requests information or reservation, the operator transfers that call to the reservation clerk. If the clerk is busy, the operator asks the caller to wait. When the clerk becomes free the operator transfers to him the call of the person who has been waiting for the longest.
Assume that arrivals and services follow *Poisson* and exponential distributions respectively. Calls arrive at a rate of ten per hour, and the reservation clerk can service a call in four minutes on the average.
(*i*) What is the average number of calls waiting to be connected to the reservation clerk ?
(*ii*) What is the average time a caller must wait before reaching the reservation clerk.
(*iii*) What is the average time for a caller to complete a call (*i.e.* waiting time plus service time) ?
[**Hint.** $\lambda = 10/60$, $\mu = 1/4$. Use formulate for L_q, W_q, W_s]
[**Ans.** $L_q = 1.33$, $W_q = 8$ min, $W_s = 12$ min.]

7. At a public telephone booth the arrivals are on the average 15 per hour. A call on the average takes 3 minutes, if there is just one phone.
(*i*) What is expected number of callers in the booth at any time,
(*ii*) For what proportion of time in the booth expected to be idle.
[**Hint.** $\lambda = 1/4$ arrival/min. $\mu = 1/3$ service/min. Use formula for L_s and $1 - \rho$.]
[**Ans.** (i) $L_s = 3$ callers, (ii) 1/4.]

8. Weavers in a *Textile* Mill arrive at a *Department Store Room* to obtain spare parts needed for keeping the looms running. The store is manned by one attendant. The average arrival rate of weavers per hour is 10 and service rate per hour is 12. Both arrival and service rate follow *Poisson* process. Determine :
(*i*) Average length of waiting line.
(ii) Averge time a machine spends in the system.
(*iii*) Percentage idle time of *Department Store Room* (attendant).
[**Ans.** (i) 6 wevers, (ii) 30 min., (ii)16.67%]

9. Consider a box office ticket window being manned by single server. Customers arrive to purchase tickets according to a *Poisson* input process with a mean rate of 30 per hour. The time required to serve a customer has an exponential distribution with a mean of 90 seconds.Determine the following : **[JNTU (MCA III) 2004, (B. Tech.) 2003]**
(*i*) Fraction of the time the server is busy. (*ii*) The average number of customer queueing for service.
(*iii*) The probability of having more than 10 customers in the system.
(*iv*) The probability that the customer has to queue for more than 3 minutes.

10. A repair shop attended by a single mechanic has an average of four customers a hour who bring small appliances for repair. The mechanic inspects them for defects and quite often can fix them right way or otherwise render a diagnosis. This takes him six minutes, on the average. Arrivals are *Poisson* and service time has the exponential distribution. You are required to :

(*i*) Find the proportion of time which the shop is empty, (*ii*) Find the probability of finding at least q customers in a shop,

(*iii*) What is the average number of customers in the system ?

(*iv*) Find the average time spend, including service.

11. People arrive at a theater ticket booth in a *Poisson* distributed arrival rate of 25 per hour.Service time is constant at 2 minutes. Calculate : (*i*) The mean number in the waiting line, (*ii*) The mean waiting time, (*iii*) The per cent of time an arrival can walk right in without having to wait. **[M.C.A. (May) 2000; C.A. (June) 91]**

12. Trucks arrive at the truck dock of a whole sale concern in a *Poisson* manner at 8 per hour. Service time distribution is approximated by negative exponential process with an average 5 minutes. Calculate :

(*i*) The number in waiting line, (*ii*) The waiting time

(*iii*) The mean number in the system (*iv*) The probability of having 6 trucks in the system.

13. In a *Bhawan Cafeteria* it was observed that there is only one bearer who takes exactly 4 minutes to serve a cup of coffee once the order has been placed with him. If the students arrive in a cafeteria at an average rate of 10 per hour, how much time one is expected to spend waiting for his tum to place the order ? **[Meerut (Maths.) 96]**

14. (a) Explain the constituents fo a single channel model.

(b) Customers arrive at the *First Class Ticket Counter* of a theatre at a rate of 12 per hour. There is one clerk serving the customers at the rate of 30 per hour :

(*i*) What is the probability that there is no customer in counter (*i.e.,* the system is idle) ?

(*ii*) What is the probability that there are more than 2 customers in the counter ?

(*iii*) What is the probability that there is no customer waiting to be served ?

(*iv*) What is the probability that a customer is being served and nobody is waiting ? **[C.A. (Nov.) 90]**

[**Hint.** Here $\lambda = 12$ per hour, $\mu = 30$ per hour, $\rho = 2/5$]

[**Ans.** (i) $P_0 = 0.6$, (ii) $P(N > 2) = \rho^3 = 0.064$, (iii) $P_0 + P_1 = 0.84$, (iv) $P_1 = 0.24$]

15. A computer manufacturing concen has sold a new brand of mini-computers to ten different organizations (one each) in a locality. The concen has employed one full time engineer to look into the complaints of malfunctioning. If the computers that malfunction arrive at the concen in a *Poisson* manner with rate 0.5 per unit time and the repair time of any particular machine is exponentially distributed with mean repair time of any particular machine is exponentially distributed with mean repair time equal to 0.5 unit, determine the steady state probability of the number of mini-computers queueing up for repairs.

16. A hospital is studying the proposal to reorganise its emergency service facility. The present arrival rate at the emergency service is 1 call every 15 minutes and the service rate is 1 call every 10 minutes. Current cost of service is Rs. 100 per hour. Each delay in service is Rs. 125. If the proposal is accepted, the service rate will become 1 call every 6 minutes. Can the organisation be justified on a strictly cost basis it the proposal increases the cost of the service by 50 %. **[Garhwal M.Sc. (Math.) 94]**

17. Patients arrive in a *Dental OPD* of general hospital in a *Poisson* manner at an average rate of 6 per hour. The doctor on average can attend to 8 patients per hour. Assuming that the service time distribution for the doctor is exponential, find: (i) Average number of patients waiting in the queue. (ii) Average time spent by a patient in the dental OPD. **[Delhi (MBA) Dec. 94]**

18. (a) Patients arrive at a clinic according to Poisson distribution at the rate of 30 patients per hour. Estimation time per patient is exponential with mean rate 20 per hour. If capacity of the clinic is unlimited, find the prob. that an arriving patient will not wait deriving the formula you use. **[Agra 98; Meerut (Math.) 98 BP]**

[**Hint.** Proceed as Ex. 8, page 242]

(b) Patients arrive at a clinic according to a Poisson distribution at the rate of 30 patients per hour. the waiting room does not accomodate more than 14 patients. Examination time per patient is exponential with mean rate 20 per hour.

(i) Find the effective arrival rate at the clinic.

(ii) What is the probability that an arriving patient will not wait ?

(iii) What is the expected waiting time until a patient is discharged from the clinic ? **[JNTU (B. Tech.) 2003]**

19. A maintenance service facility has Poisson arrival rates, negative exponential service times, and operates on a first-come first-served queue discipline. Breakdowns occur on an average of three per day with a range of zero to eight. The maintenance crew can service on an average six machines per day with a range from zero to seven. Find the

(i) Utilisation factor of the service facility,

(ii) mean time in the system,

(iii) mean number in the system in breakdown or repair,

(iv) mean waiting time in the queue,

(v) probability of finding two machines in the system.

(vi) expected number in the queue. **[M.G. (M.B.A.) Dec. 98]**

[**Ans.** (i) $\rho = 50\%$ (ii) $W_s = \frac{1}{3}$ per day (iii) $L_s = 1$ machine, (iv) $W_1 = \frac{1}{6}$ per day, (v) $P_2 = 0{\cdot}125$, (vi) $L_q = \frac{1}{2}$ machine]

20. A bank has one drive-in-counter. It is estimated that cars arrive according to Poisson distribution at the rate of 2 every 5 minutes and that there is enough space to accommodate a line of 10 cars. Other arriving cars can wait outside this space, if necessary. It takes 1·5 minutes on an average to serve a customer, but the service time actually varies according to an exponential distribution. You are required to find :

(i) the proportion of time the facility remains idle;
(ii) the expected number of customers waiting but currently not being served at a particular point of time;
(iii) the expected time a customer spends in the system, and
(iv) the probability that the waiting line will exceed the capacity of the space leading to the drive-in counter. **[C.A. (May) 97]**

[**Hint.** $\lambda = \frac{2 \times 60}{5}$, $\mu = \frac{60}{1 \cdot 5}$

Ans. (i) $P_0 = 0{\cdot}4$, *i.e.*, 40% (ii) $L_q = 0{\cdot}9$, (iii) $W_s = 3{\cdot}75$ minutes (iv) $P(n \geq 11) = (\lambda/\mu)^{11} = 0{\cdot}036$]

21. Arrivals of customers to a payment counter (only one) in a bank follow Poisson distribution with an average of 10 per hour. The service time follows negative exponential distribution with an average of 4 minutes.

(i) What is the average number of customers in the queue ?
(ii) The bank will open one more counter when the waiting time of a customer is at least 10 minutes. By how much the flow of arrivals should increase in order to justify the second counter ? **[Jammu (M.B.A.) 97]**

[**Hint.** λ = 10/hour and μ = 15/hour.

Ans. (i) $L_q = 4/3$ (ii) $W_q' = \frac{\lambda'}{\mu(\mu - \lambda')} \Rightarrow \frac{10}{60} = \frac{\lambda'}{15(15 - \lambda')}$ or $\lambda' = 10{\cdot}7$.

Thus arrival rate should increase by 0·7 hour.]

22. A refinery distributes its products by trucks loaded at the loading dock. There are both company trucks and independent distributors' trucks. The independent distributors complained that sometimes they must wait in line and thus lose money paying for truck and driver that is only waiting. They have asked the refinery either to put in a second loading dock or to discount price equivalent to the waiting time. Extra loading dock costs Rs. 100 per day wheres the waiting time for the independent firms costs Rs. 25 per hour.

The following data of arrival and service at the dock are available and they are Poisson distributed :

Average arrival rate of all trucks = 2 per hour.
Average service rate = 3 per hour.

Thirty per cent of all trucks are from independent distributors.

(i) What is the expected cost of waiting time of the independent distributors ?
(ii) Is it advantageous to decide in favour of a second loading dock to ward off the complaints ? **[Bombay (M.M.S.) 95]**

23. A single crew is provided for unloading and/or loading each truck that arrive at the loading deck of a warehouse. These trucks arrive according to a Poisson input process at a mean rate of one hour. The time required by a crew to unload and/or laod a truck has an exponential distribution (regardless of the crew size). The expected time required by a one-man crew would be two hours.

The cost of providing each additional member of the crew is Rs. 10 per hour. The cost that is attributable to having a truck not in use (*i.e.*, a truck standing at the loading deck) is estimated to be Rs. 40 per hour.

Assume that the mean service rate of the crew is proportional to its size. What should be the size in order to minimize the expected total cost per hour ? **[Delhi (M.B.A.) 96]**

24. A factory operates for 8 hours everyday and has 240 working days in the year. It buys a large number of small machines which can be serviced by its maintenance engineer at a cost of Rs. 5 per hour for the labour and spare parts. The machines can, alternatively, be serviced by the supplier at an annual contact price of Rs. 25,000 including labour and spare parts needed. The supplier undertakes to send a repairman as soon as a call is made but in no case more than one repairman is sent. The service time of the maintenance engineer and the supplier's repairman are both exponentially distributed with respective means of 1·7 and 1·5 days. The machine breakdowns occur randomly and follow Poisson distribution , with an average of 2 in 5 days. Each hour that a machine is out of order, it costs the company Rs. 10. Which servicing alternative would you advise it to opt for ? **[Allahabad (M.B.A.) 97]**

[**Hint.** *Alternative* 1 : *Serviced by company's Engineer*

$\lambda = 2/5$ mach/day, $\mu = 1/1{\cdot}7$ mach/day

Total cost = cost of machine × + cost of labour and spare parts

$$= \left(\frac{\lambda}{\mu - \lambda}\right)(240 \times 8) \times 10 + \left(\frac{\lambda}{\mu - \lambda}\right)(240 \times 8) \times 5$$

= Rs. 61,200

Alternative 2. Mantenance by supplier :

$\lambda = \frac{2}{3}$ mach/day, $\mu = \frac{1}{1 \cdot 5}$ mach/day

Total cost = $\frac{3}{2} \times 1920 \times 10 + 25{,}000$ = Rs. 53,800

∴ second alternative is the best one.]

25. A car park has space to accomodate 40 cars. The arrival of cars is Poisson at a mean rate of 2 per minute. The length of time each car spends in the car rank has negative exponetial distribution with mean of 30 minutes ?

(a) What is the probability of a newly arriving customer finding the car park full ?

(b) How many cars are in the car park on average ?

(c) What is the probability of having zero cars in the car park space. **[JNTU (B. Tech.) 2003]**

26. A company currently has two tools cribs, each having a single clerk, in its manufacturing area. One tool crib handles only the tools for the heavy machinery, while the second one handles all other tools. It is observed that for each tool crib the arrivals follow a Poisson distribution with a mean of 20 per hour, and the service time distribution is negative exponential with a mean of 2 minutes.
The tool manager feels that, if tool cribs are combined in such a way that either clerk can handle any kind of tool as demand arises, the system would be more efficient and the waiting problem could be reduced to some extent. It is believed that the mean arrival rate at the two tool cribs will be 40 per hour; while the service time will remain unchanged. Compare the existing system with the one proposed with respect to the total expected number of machines at the tool crib(s), the expected waiting time including service time for each mechanic and probability that he hasa to wait for service. **[Delhi (M.B.A.) March 99]**

27. The men's department of a large store employs one tailor for customer fittings. The number of customers requiring fittings appears to follow a poisson distribution with mean arrival rate 24 per hour. Customers are fitted on a first-come, first-served basis, and they are always willing to wait for the tailors service, because alterations are free. The time it takes to fit a customer appears to be exponentially distributed, with a mean of 2 min.
(i) What is the average number of customers in the fitting room ?
(ii) How much time a customer is expected to spend in the fitting room ?
(iii) What percentage of the time is the tailor idle ? **[VTU (BE Mech.) 2002]**

28. Problems arrive at a computing center in Poisson fashion with a mean arrival rate of 25% per hour. The average computing job requires 2 minutes of terminal of time. Calculate the following :
(a) average no. of problems waiting for the computer.
(b) the percentage of times on arrival can walk right in without having to wait. **[JNTU (MCA III) 2004]**

29. A firm is engaged in both shipping and receiving activities. The management is always interested in improving the efficiency by new innovations in loading and unloading procedures. The arrival distribution of trucks is found to be Poission with arrival rate of two trucks per hour. The service time distribution is exponential with unloading rate of three trucks per hour. Find the following :
(a) Average no. of trucks in waiting time (b) Average waiting time of trucks in line.
(c) The prob. that the loading and unloading dock and workers will be idle.
(d) What reductions in waiting time are possible if loading and unloading is standarized ?
(d) What reductions are possible if lift trucks are used. **[JNTU (Mech. & Prod.) Main 2004]**

23.13. Model II (A). General Erlang Queueing Model (Birth-Death Process) [Agra 98]

(a) To obtain the system of steady state equations

Let

$$\left.\begin{array}{l}\text{arrival rate } \lambda = \lambda_n \\ \text{service tate } \mu = \mu_n\end{array}\right\} \quad \text{[depending upon } n\text{]}$$

Then, by the same arguments as for equations (23.51) and (23.53),

$$P_n(t + \Delta t) = P_n(t)\,[1 - (\lambda_n + \mu_n)\,\Delta t] + P_{n-1}(t)\,\lambda_{n-1}\Delta t + P_{n+1}(t)\,\mu_{n+1}\,\Delta t + O(\Delta t),\ n > 0; \quad \text{...(23.73)}$$

and $$P_0(t + \Delta t) = P_0(t)\,[1 - \lambda_0\,\Delta t] + P_1(t)\,\mu_1\,\Delta t + O(\Delta t),\ n = 0. \quad \text{...(23.74)}$$

Now dividing (23.73) and (23.74) by Δt, taking limits as $\Delta t \to 0$ and following the same procedure as in Model I, obtain

$$\frac{dP_n(t)}{dt} = -(\lambda_n + \mu_n)\,P_n(t) + \lambda_{n-1}\,P_{n-1}(t) + \mu_{n+1}\,P_{n+1}(t) \quad \text{...(23.75)}$$

and $$\frac{dP_0(t)}{dt} = -\lambda_0\,P_o(t) + \mu_1\,P_1(t) \text{ respectivcly.} \quad \text{...(23.76)}$$

The equations (23.75) and (23.76) are differential-difference equations which could be solved if a set of initial values $P_0(0)$, $P_1(0)$, ..., is given. Such a system of cquations can be solved if the time dependent solution is required. But, for many problems it suffices to look at the steady state solution.

In the case of steady state,

$$P_n'(t) = 0 \text{ and } P_0'(t) = 0.$$

So the equations (23.75) and (23.76) become,

$$0 = -(\lambda_n + \mu_n)\,P_n + \lambda_{n-1}\,P_{n-1} + \mu_{n+1}\,P_{n+1},\ n > 0 \quad \text{...(23.77)}$$

and $$0 = -\lambda_0\,P_0 + \mu_1\,P_1,\ n = 0. \quad \text{...(23.78)}$$

The equations (23.77) and (23.78) constitute the system of steady state difference equations for this model.

(b) To solve the system of difference equations **[Agra 98]**

Since $P_0 = P_0$

$$P_1 = \frac{\lambda_0}{\mu_1} P_0 \quad \text{[from equation (23.78)]}$$

$$P_2 = \frac{\lambda_1}{\mu_2} P_1 = \frac{\lambda_1 \lambda_0}{\mu_2 \mu_1} P_0 \quad \text{[from letting } n = 1 \text{ in equation (23.77) and substituting for } P_1\text{]}$$

$$P_3 = \frac{\lambda_2}{\mu_3} P_2 = \frac{\lambda_2 \lambda_1 \lambda_0}{\mu_3 \mu_2 \mu_1} P_0 \quad \text{[putting } n = 2 \text{ in equation (23.77)]}$$

$$\ldots \quad \ldots \quad \ldots$$

$$P_n = \frac{\lambda_{n-1} \lambda_{n-2} \ldots \lambda_0}{\mu_n \mu_{n-1} \ldots \mu_1} P_0 \quad [\text{for } n \geq 1] \qquad \ldots(23.79)$$

Now, in order to find P_0, use the fact that

$$\sum_{n=0}^{\infty} P_n = 1. \text{ or } P_0 + P_1 + P_2 + \ldots \infty = 1 \text{ or } P_0 \left[1 + \frac{\lambda_0}{\mu_1} + \frac{\lambda_0 \lambda_1}{\mu_1 \mu_2} + \ldots \infty\right] = 1 \text{ or } P_0 = 1/S,$$

where

$$S = 1 + \frac{\lambda_0}{\mu_1} + \frac{\lambda_0 \lambda_1}{\mu_1 \mu_2} + \ldots \infty. \qquad \ldots(23.80)$$

Note. The series $S = 1 + \frac{\lambda_0}{\mu_1} + \frac{\lambda_0 \lambda_1}{\mu_1 \mu_2} + \ldots \infty$ is summable and meaningful only when it is convergent.

The result obtained above is a general one and by suitably defining μ_n and λ_n many interesting cases could be studied. Now three particular cases may arise :

Case 1. $(\lambda_n = \lambda, \mu_n = \mu)$

In this case, the series S becomes

$$S = 1 + \frac{\lambda}{\mu} + \left(\frac{\lambda}{\mu}\right)^2 + \ldots \infty = \frac{1}{1 - \lambda/\mu} \qquad (\text{when } \lambda/\mu < 1)$$

Therefore, from equations (23.80) and (23.79),

$$P_0 = \frac{1}{S} = 1 - \frac{\lambda}{\mu} \quad \text{and} \quad P_n = \left(\frac{\lambda}{\mu}\right)^n \left(1 - \frac{\lambda}{\mu}\right)$$

Here, it is observed that this is exactly the case of Model I.

Case 2. $\left(\lambda_n = \frac{\lambda}{n+1}, \mu = \mu\right)$

The case, in which the arrival rate λ_n depends upon n inversely and the rate of service μ_n is independent of n, is called the case of *"Queue with Discouragement"*.

In this case, the series S becomes

$$S = 1 + \frac{\lambda}{\mu} + \frac{\lambda^2}{2\mu^2} + \frac{\lambda^3}{2.3\mu^3} + \ldots \infty = 1 + \rho + \frac{\rho^2}{2\,!} + \frac{\rho^3}{3\,!} + \ldots \infty = e^{\rho} \qquad (\rho = \lambda/\mu)$$

The equation (23.80) gives $P_0 = 1/S = e^{-\rho}$.

Also,

$$P_1 = \frac{\lambda}{\mu} P_0 = \rho\, e^{-\rho},$$

$$P_2 = \frac{\lambda^2}{2\mu^2} P_0 = \frac{\rho^2 e^{-\rho}}{2\,!},$$

$$\ldots \quad \ldots \quad \ldots$$

$$P_n = \frac{\lambda^n}{n\,!\,\mu^n} P_0 = \frac{\rho^n e^{-\rho}}{n\,!} \text{ for all } n = 0, 1, \ldots, \infty.$$

It is observed in this case that P_n follows the Poisson distribution, where $\lambda/\mu = \rho$ is constant, however $\rho > 1$ or $\rho < 1$ but must be finite. Since, the series S is convergent and hence summable in both the cases.

Case 3. $(\lambda_n = \lambda$ **and** $\mu = n\mu)$, *i.e.*, **the case of infinite number of stations.**

In this case, the arrival rate λ_n does not depend upon n, but the service rate μ_n increases as n increases. Here, assume that there are infinite (variable) number of service stations. The word *'infinite'* means that the service stations are available to each arrival. But it does not mean that all the infinite service stations will remain busy every time. In other words, it means that if n customers arrive, then n service stations will be available for all $n = 0, 1, 2, \ldots\infty$. Obviously, no queue will form in this case because each arrival will immediately enter the service facility. For example, in everyday life, it is observed that the telephone (service stations) are always available to all the arriving persons.

In this case, the series S becomes

$$S = 1 + \frac{\lambda}{\mu} + \frac{\lambda^2}{2.1\mu^2} + \frac{\lambda^3}{3.2.1\mu^3} + \ldots\infty = 1 + \rho + \frac{\rho^2}{2\,!} + \frac{\rho^3}{3\,!} + \ldots\infty = e^{\rho}, \text{ where } \rho = \lambda/\mu.$$

Therefore, $\quad P_0 = e^{-\rho}$ and $P_n = e^{-\rho}\,\rho^n/n\,!$

which again follows the *Poisson* distribution law.

23.13-1. Model II (B). (*M* I *M* I 1) : (∞ I *SIRO*)

This model is actually the same as *Model* I, except that the service discipline follows the *Service In Random Order* (*SIRO*) rule in place of *FCFS*-rule. Since the derivation of P_n in *Model* I does not depend on any specific queue discipline, we must have

$$P_n = (1 - \rho).\rho^n, n \geq 0$$

for the *SIRO*-rule case also.

Consequently, the average number of customers in the system (L_s) will remain the same, whether the queue discipline follows the *SIRO*-rule or *FCFS*-rule. In fact, L_s will not change provided P_n remains unchanged for any queue discipline. Thus $W_s = L_s/\lambda$ under the *SIRO*-rule is also same as under the *FCFS*-rule and it is given by $W_s = 1/(\mu - \lambda)$.

Moreover, this result can be extended to *any queue discipline* so long as P_n remains unchanged. In particular, the result is applicable to the three most common queue disciplines : *FCFS, LCFS* and *SIRO*. These queue disciplines differ only in the distribution of waiting time where the probabilities of long and short waiting times change depending upon the queue discipline used. So we can use the *GD* (*General Discipline*) to represent *FCFS, LCFS* and *SIRO*, whenever the waiting time distribution is not required.

Q. 1. Deduce the difference equations for the queueing model M I M I 1 : (FCFS/∞/∞) with arrival and service rates dependent on system size. Obtain steady state solution. Deduce also the solution for the following special cases :
(i) Queues with discouragement, (ii) Queues with ample servers. **[Delhi MA/M.Sc. (OR.) 92, 90]**

2. Derive differential-difference equations for a generalized birth-death queueing model. Obtain steady-state distribution of the system size. **[Delhi MA/M.Sc (State.) 95]**

23.13-2. Illustrative Examples on Model II

Example 26. *Problems arrive at a computing centre in a Poisson fashion at an average rate of five per day. The rules of the computing centre are that any man waiting to get his problem solved must aid the man whose problem is being solved. If the time to solve a problem with one man has an exponential distribution with mean time of ⅓ day, and if the average solving time is inversely proportional to the number of people working on the problem, approximate the expected time in the centre for a person entering the line.* **[Rohil. 92]**

Solution. Here $\lambda = 5$ problems/day

$\mu = 3$ problems/day (mean service rate with one unsolved problem)

Then, the expected number of persons working at any specified instant is :

$$L_s = \sum_{n=0}^{\infty} nP_n \text{, where } P_n = \frac{1}{n\,!}\left(\frac{\lambda}{\mu}\right)^n e^{-\lambda/\mu} \text{ [see **Case II**]}$$

$$= \sum_{n=0}^{\infty} n\frac{1}{n\,!}\left(\frac{\lambda}{\mu}\right)^n e^{-\lambda/\mu} \qquad = e^{-\lambda/\mu}\sum_{n=0}^{\infty}\frac{n}{n\,!}\cdot\left(\frac{\lambda}{\mu}\right)^n$$

$$= e^{-\lambda/\mu}\left\{\frac{\lambda}{\mu} + \frac{2}{2!}\left(\frac{\lambda}{\mu}\right)^2 + \frac{3}{3!}\left(\frac{\lambda}{\mu}\right)^3 + \dots\infty\right\}$$

$$= e^{-\lambda/\mu} \cdot \frac{\lambda}{\mu}\left\{1 + \frac{\lambda}{\mu} + \frac{1}{2!}\cdot\left(\frac{\lambda}{\mu}\right)^2 + \dots\right\}$$

$$= e^{-\lambda/\mu} \cdot \frac{\lambda}{\mu} e^{\lambda/\mu} = \lambda/\mu .$$

Substituting the values for λ and μ, $L_S = 5/3$ persons.

Now, the average solving time which is inversely proportional to the number of people working on the problem is : $1/5$ day/problem.

Therefore, expected time for a person entering the line is

$$= 1/5 \times L_S \text{ days} = 1/5 \times 5/3 \times 24 \text{ hours} = 8 \text{ hours.}$$ **Ans.**

Example 27. *A shipping company has a single unloading berth with ships arriving in Poisson fashion at an average rate of three per day. The unloading time distribution for a ship with the unloading crews is found to be exponential with average unloading time $1/2$ in days. The company has a large labour supply without regular working hours and to avoid long waiting lines the company has a policy of using as many unloading crews as there are ships waiting in line or being unloaded. Under these conditions,* find

(i) *the average number of unloading crews working at any time,* and

(ii) *the probability that more than four crews will be needed.*

Solution. Here,

$\lambda = 3$ ships per day

$\mu = 2$ ships per day (mean service rate with one unloading crew)

(i) Average number of unloading crews working at any specified instant is :

$$L_s = \sum_{n=0}^{\infty} n P_n = \sum_{n=0}^{\infty} n \frac{e^{-\lambda/\mu}}{n!}\left(\frac{\lambda}{\mu}\right)^n \left[\text{since } P_n = \frac{e^{-\lambda/\mu}}{n!}\left(\frac{\lambda}{\mu}\right)^n\right]$$

$= \lambda/\mu = 3/2$ crews. (*see example 26 above*) **Ans.**

(ii) The probability that more than 4 crews will be needed is the same as the probability that there are at least five ships in the system at any specified instant which is given by

$$\sum_{n=5}^{\infty} P_n = \sum_{n=0}^{\infty} P_n - \sum_{n=0}^{4} P_n = 1 - [P_0 + P_1 + P_2 + P_3 + P_4]$$

$$= 1 - \left[e^{-\lambda/\mu} + \left(\frac{\lambda}{\mu}\right)\frac{e^{-\lambda/\mu}}{1!} + \left(\frac{\lambda}{\mu}\right)^2\frac{e^{-\lambda/\mu}}{2!} + \left(\frac{\lambda}{\mu}\right)^3\frac{e^{-\lambda/\mu}}{3!} + \left(\frac{\lambda}{\mu}\right)^4\frac{e^{-\lambda/\mu}}{4!}\right]$$

$$= 1 - e^{-\lambda/\mu}\left[1 + \frac{\lambda}{\mu} + \frac{(\lambda/\mu)^2}{2!} + \frac{(\lambda/\mu)^3}{3!} + \frac{(\lambda/\mu)^4}{4!}\right]$$

Now putting values for λ and μ, and simplifying, we get

$$\sum_{n=5}^{\infty} P_n = 0.019.$$ **Ans.**

23.14. MODEL III. $(M \mid M \mid 1) : (N \mid FCFS)$

Up to this stage, only two models are discussed in which the capacity of the system is infinite. Now consider the case where the capacity of the system is limited, say N. In fact the number of arrivals will not exceed the N in any case.

The physical interpretation for this model may be either :

(i) that there is only room (capacity) for N units in the system (as in a packing lot),

or *(ii)* that the arriving customers will *go for their service elsewhere* permanently, if the waiting line is too long ($\leq N$).

(a) To obtain steady state difference equations. The simplest way of starting this is to treat the model as a special case of *Model* II, where

$$\lambda_n = \begin{cases} \lambda, & n = 0, 1, 2, 3, \dots, N-1 \\ 0, & n \geq N \end{cases} \qquad \dots(23.81)$$

and $\mu_n = \mu$ for $n = 1, 2, 3, \ldots$ ...(23.82)

Now, following the similar arguments as given for equations (23.53) and (23.51) in Model I, we obtain

$P_0(t+\Delta t) = P_0(t)[1 - \lambda\Delta t] + P_1(t)\mu\Delta t + O(\Delta t)$, for $n = 0$, ...(23.83)

$P_n(t+\Delta t) = P_n(t)[1 - (\lambda+\mu)\Delta t] + P_{n-1}(t)\lambda\Delta t + P_{n+1}(t)\mu\Delta t + O(\Delta t)$,
for $n = 1, 2, \ldots, N-1$, ...(23.84)

and $P_N(t+\Delta t) = P_N(t)[1 - (0+\mu)\Delta t] + P_{N-1}(t)\lambda\Delta t + 0 \times \mu\Delta t + O(\Delta t)$
$= P_N(t)[1 - \mu\Delta t] + P_{N-1}(t)\lambda\Delta t + O(\Delta t)$ for $n = N$, $P_{N+1}(t) = 0$, $\lambda = 0$ ...(23.85)

Now dividing equation (23.83), (23.84), and (23.85) by Δt, and taking limit as $\Delta t \to 0$, these equations transform into

$P_0'(t) = -\lambda P_0(t) + \mu P_1(t)$ for $n = 0$ ...(23.83a)

$P_n'(t) = -(\lambda+\mu)P_n(t) + \lambda P_{n-1}(t) + \mu P_{n+1}(t)$ for $n = 1, 2, \ldots, N-1$, ...(23.84a)

and $P_N'(t) = -\mu P_N(t) + \lambda P_{N-1}(t)$ for $n = N$. ...(23.85a)

In the case of steady state, when $t \to \infty$, $P_n(t) \to P_n$ (independent of t) and hence $P_n'(t) \to 0$. So the system of steady state difference equations is given by

$0 = -\lambda P_0 + \mu P_1$, for $n = 0$ ...(23.83b)

$0 = -(\lambda+\mu)P_n + \lambda P_{n-1} + \mu P_{n+1}$, for $n = 1, 2, \ldots, N-1$, ...(23.84b)

$0 = -\mu P_N + \lambda P_{N-1}$, for $n = N$. ...(23.85b)

(b) To solve the system of difference equations (23.83b), (23.84b) and (23.85b).

Here

$P_0 = P_0$ (initially)

$P_1 = \frac{\lambda}{\mu} P_0$ [from (23.83b)]

$P_2 = \left(\frac{\lambda}{\mu}\right)^2 P_0$ [put $n = 1$ in (23.84b) and substitute value of P_1].

Similarly,

$P_3 = \left(\frac{\lambda}{\mu}\right)^3 P_0$

... ...

$P_n = \left(\frac{\lambda}{\mu}\right)^n P_0, n < N$ ...(23.86)

$P_N = \left(\frac{\lambda}{\mu}\right)^N P_0$, $n = N$ (because $P_N = (\lambda/\mu) P_{N-1}$ and P_{N-1} follows the rule for which $n = 1, 2, \ldots, N-1$).

$P_{N+1} = 0, n > N$.

Now, in order to find P_0, use the fact that :

$$\sum_{n=0}^{N} P_n = 1$$

or $P_0[1 + (\lambda/\mu) + (\lambda/\mu)^2 + \ldots (\lambda/\mu)^N] = 1$ or $P_0\left[\frac{1-\rho^{N+1}}{1-\rho}\right] = 1$, where $\rho = \lambda/\mu$

or $P_0 = \frac{1-\rho}{1-\rho^{N+1}}$. ...(23.87)

Substituting the value of P_0 in (23.86),

$P_n = \left(\frac{1-\rho}{1-\rho^{N+1}}\right)\rho^n$, for $n = 0, 1, 2, \ldots, N$. ...(23.88)

Thus, the result (23.87) and (23.88) give the required solution for this model which do not require $\lambda < \mu$. That is, in this case, ρ may be greater than 1 also.

(c) Measures of Model III.

(i) $L_s = \sum_{n=0}^{N} nP_n = \sum_{n=0}^{N} n\left(\frac{1-\rho}{1-\rho^{N+1}}\right)\rho^N$

or $$L_s = \frac{1-\rho}{1-\rho^{N+1}} \sum_{n=0}^{N} n\rho^n = P_0 \sum_{n=0}^{N} n\,\rho^n. \qquad ...(23.89)$$

Now, four relationships (23.69), (23.70), (23.71) and (23.72) give :

(ii) $L_q = L_s - \lambda/\mu$ ...(23.90)

(iii) $W_s = L_s/\lambda$, and ...(23.91)

(iv) $W_q = W_s - 1/\mu = L_q/\lambda$. ...(23.92)

Q. 1. Obtain the steady state difference equations for the queueing model $(M \mid M \mid 1) : (N \mid FCFS)$ in usual notations and solve them for P_0 and P_1. Also find the mean queue length for this system. **[Meerut (Stat.) 98; Garhwal (Stat.) 92]**

2. For the $(M \mid M \mid 1) : (FCFS, K)$ queueing model, show that the steady state probability, p_n is given by

$$p_n = \rho^n \frac{1-\rho}{1-\rho^{K+1}}, \; 0 \le n \le K.$$

Also obtain expected number of units in the queue and system separately.

3. For the model $(M \mid M \mid 1) : (N \mid FCFS)$ where the notations have their usual meanings, find the following :

(i) The average number of customers in the system. (ii) Average queue length.

4. Explain $(M \mid M \mid 1) : (N \mid FCFS)$ system and solve it in steady state. **[Garhwal M.Sc. (Stat.) 96, 95, 93]**

23.14–1. Illustrative Examples on Model III

Example 28. *In Example 5 of sec. 23.12–2, if we assume that the line capacity of yard is to admit of 9 trains only (there being 10 lines, one of which is ear marked for the shunting engine to reverse itself from the crest of the hump to the rare of the train). Calculate the following on the assumption that 30 trains, on average, are received in the yard :*

(a) *the probability that the yard is empty,* (b) *average queue length.*

Solution. As already computed in *Example* 5, **section 23.12–2,** $\rho = 0.75$.

(a)The probability 'that the queue size is zero' is given by

$$P_0 = (1-\rho)/(1-\rho^{N+1})$$

But given that $N = 9$ so,

$$P_0 = \frac{1-0.75}{1-(0.75)^{10}} = \frac{0.25}{0.90} = 0.28. \qquad \textbf{Ans.}$$

(b) Average queue length is given by the formula

$$L_s = \left(\frac{1-\rho}{1-\rho^{N+1}}\right) \sum_{n=0}^{N} n\rho^n$$

$$\therefore \quad L_s = \frac{1-0.75}{1-(0.75)^{10}} \sum_{n=0}^{9} n\,(0.75)^n = 0.28 \times 9.58 = 2.79, \text{ say 3 trains.} \qquad \textbf{Ans.}$$

Example 29. *If for a period of 2 hours in a day (8—10 A.M.) trains arrive at the yard every 20 minutes but the service time continues to remain 36 minutes, then calculate for this period :*

(a) *the probability that the yard is empty,* **(b)** *average queue length, on the assumption that the line capacity of the yard is limited to 4 trains only.*

Solution. Here, $\rho = 36/20 = 1.8$ (which is greater than 1) and $N = 4$.

Thus, we obtain

(a) $P_0 = \dfrac{\rho - 1}{\rho^5 - 1} = 0.04$

(b) average queue size $= P_0 \sum_{n=0}^{4} n\,\rho^n$, $= .04\,[\rho + 2\rho^2 + 3\rho^3 + 4\rho^4]$, $= .04 \times 72.0 = 2.9$, say 3 trains. **Ans.**

EXAMINATION PROBLEMS (ON MODEL III)

1. Discuss the stationary state of the queue system $(M \mid M \mid 1) : (N \mid FCFS)$.

A car park contains 5 cars. The arrival of cars is Poisson at a mean rate of 10 per hour. The length of time each car spends in the car park has negative exponential distribution with mean of 5 hours. How many cars are in the car park on average and what is the probability of a newly arriving customer finding the car park full and having to park his car elsewhere ?

[**Hint.** Here $N = 5$, $\lambda = 10/60$, $\mu = 1/2 \times 60$, $\rho = \lambda/\mu = 20$. Find $P_0 = \frac{1-\rho}{1-\rho^{N+1}}$, $L_s = P_0 \sum_{n=0}^{N} n\rho^n$.]

2. A barber shop has space to accommodate only 10 customers. He can serve only one person at a time. If a customer comes to his shop and finds it full he goes to the next shop. Customers randomly arrive at an average rate $\lambda = 10$ per hour and the barber's service time is negative exponential with an average of $1/\mu = 5$ minutes per customer.
   (i) Write recurrence relations for the steady state queueing system (*FCFS*) for above.
   (ii) Determine P_0 and P_n, probability of having 0 and *n*-customers respectively in the shop.

   [**Hint.** Here $N = 10$, $\lambda = 10/60$, $\mu = 1/5$, $\rho = \lambda/\mu = 5/6$. Find $P_0 = \frac{1-\rho}{1-'rho^{N+1}}$, $P_n = P_0\rho^n$.]

3. Patients arrive at a clinic according to a Poisson distribution at the rate of 30 patients per hour. The waiting room does not accommodate more than 14 patients. Examination time per patient is exponential with mean rate 20 per hour.
   (i) Find the effective arrival rate at the clinic.
   (ii) What is the probability that an arriving patient will not wait ? Will he find a vacant seat in the room ?
   (iii) What is the expected waiting time until a patient is discharged from the clinic ?

   [**Hint.** Here $N = 14$, $\lambda = 30/60$, $\mu = 20/60$, $\rho = 2/3$. Find P_n, P_0 and $W_s = \frac{P_0}{\lambda} \sum_{n=0}^{N} n\rho^n$.] **[Meerut (MCA) 2000]**

4. Customers arrive at a one-window derive-in-counter according to a Poisson distribution with mean 10 per hour. Service time per customer is exponential with mean 5 minutes. The car space in front of the window, including that for the serviced can accomodate a maximum of 3 cars. Other cars can wait outside this space.
   (a) What is the probability that an arriving customer can drive directly to the space in front of the window ?
   (b) What is the probability that an arriving customer will have to wait outside the indicated space ?
   (c) How long is an arriving customer expected to wait before starting service ?
   (d) How, many spaces should be provided in front of the window so that all the arriving customers can wait in front of the window at least 20% of the time. **[JNTU (Mech. & Prod.) 2004; Meerut (MCA) 2000]**

   [**Hint.** $\lambda = 10$, $\mu = 60/5 = 12$, $P_0 = 1 - (\lambda/\mu) = 1/6$, (a) $P_0 + P_1 + P_2 = [1 + \lambda/\mu + (\lambda/\mu)^2]\, P_0 = 0.42$,
   (b) $1 - (P_0 + P_1 + P_2 + P_3) = 1 - 0.42 - P_3 = 0.58 - (\lambda/\mu)^3 P_0 + 0.48$. (c) $W_q = \lambda/\{\mu(\mu - \lambda)\} = 0.417$,
   (d) $P_0 + P_1 = 0.30$. Hence there should be at least one car space for waiting at least 20% of the time.]

5. A stenographer has 5 person for whom she performs stenographic work. Arrival rate is Poisson and service times are exponential. Average arrival rate is 4 per hour with an average service time of 10 minutes. Cost of waiting is Rs. 8 per hour while the cost of servicing is Rs. 2.50 each. Calculate :
   (i) the average waiting time of an arrival,
   (ii) the average length of the waiting line,
   (iii) the average time which an arrival spends in the system, and
   (iv) the minimum cost service rate.

   [**Ans.** (i) 12.4 min., (ii) 0.79 = one stenographer, (iii) 22.4 min.]

23.15. MODEL IV (A) : (*M* | *M* | *S*) : (∞ | *FCFS*)

In this more realistic queueing system the customers arrive in a Poisson fashion with mean arrival rate λ. There are s (fixed) number of counters (service stations) arranged in parallel, and a customer can go to any of the free counters for his service, where the service time at each counter is *identical* and follows the same exponential distribution law. The mean service rate per busy server is μ. Therefore, over all service rate, when there are n units in the system, may be obtained in the following two situations :

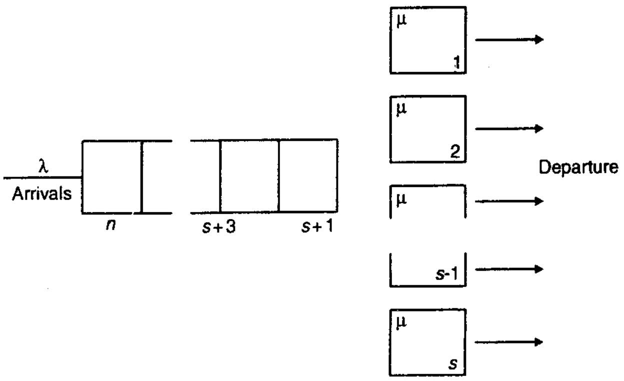


Fig. 23.19. Here the number of equivalent parallel service stations is *s*, and max. length of queue is *n* = *s*.

(i) If $n \le s$, all the customers may be served simultaneously. There will be no queue,

$(s - n)$ number of servers may remain idle, and then

$$\mu_n = n\mu \,,\ n = 0, 1, 2, \dots, s \,;$$

(ii) If **n ≥ s,** all the servers are busy, maximum number of customers waiting in queue will be $(n - s)$, then $\mu_n = s\mu$.

Now, this model may also be considered as a special case of Model II with

$$\lambda_n = \lambda \text{ (for } n = 0, 1, 2, \dots)$$

$$\mu_n = \begin{cases} n\mu \text{ (for } n = 0, 1, 2, \dots, s) \\ s\mu \text{ (for } n \geq s) \end{cases}$$

Consequently, the steady state results are obtained as follows :

(a) To obtain the system of steady state equations.

Following the similar arguments as for equations (23.53) and (23.51) in Model I, we get

$$P_0(t + \Delta t) = P_0(t)\,[1 - \lambda\Delta t\,] + P_1(t)\,\mu\Delta t + O(\Delta t)\,,\ \text{for } n = 0, \qquad \text{...(23.93)}$$

$$P_n(t + \Delta t) = P_n(t)\,[1 - (\lambda + n\mu)\,\Delta t\,] + P_{n-1}(t)\,.\,\lambda\Delta t + P_{n+1}(t)\,(n + 1)\,\mu\Delta t + O\,(\Delta t),$$
$$\text{for } n = 1, 2, 3, \dots, s - 1\,; \qquad \text{...(23.94)}$$

and $$P_n\,(t + \Delta t) = P_n\,(t)\,[1 - (\lambda + s\mu)\,\Delta t\,] + P_{n-1}\,(t)\,\lambda\Delta t + P_{n+1}\,(t)\,s\mu\Delta t + O\,(\Delta t)\,,$$
$$\text{for } n = s, s + 1, s + 2, \dots \qquad \text{...(23.95)}$$

Now dividing these equations by Δt and taking limit as $\Delta t \to 0$, the governing *differential-difference equations* are

$$P_0'(t) = -\,\lambda P_0(t) + \mu P_1(t), \quad \text{for } n = 0 \qquad \text{...(23.93a)}$$

$$P_n'(t) = -\,(\lambda + n\mu)\,P_n(t) + \lambda P_{n-1}(t) + (n + 1)\,\mu P_{n+1}(t)\,, \quad \text{for } n = 1, 2, \dots, s - 1 \qquad \text{...(23.94a)}$$

$$P_n'(t) = -\,(\lambda + s\mu)\,P_n(t) + \lambda P_{n-1}(t) + s\mu P_{n+1}(t), \text{ for } n \geq s. \qquad \text{...(23.95a)}$$

Considering the case of steady state, *i.e.* when $t \to \infty$, $P_n\,(t) \to P_n$ (independent of t) and hence $P_n'\,(t) \to 0$ for all n , above equations become

$$0 = -\,\lambda P_0 + \mu P_1\,,\ \text{for } n = 0 \qquad \text{...(23.93b)}$$

$$0 = -\,(\lambda + n\mu)\,P_n + \lambda P_{n-1} + (n + 1)\,\mu P_{n+1}\,,\ \text{for } 0 < n < s \qquad \text{...(23.94b)}$$

$$0 = -\,(\lambda + s\mu)\,P_n + \lambda P_{n-1} + s\mu P_{n+1}\,,\ \text{for } n \geq s. \qquad \text{...(23.95b)}$$

This is the system of steady state *difference equations.*

(b) To solve the system of difference equations (23.93b), (23.94b) and (23.95b).

Here $\quad P_0 = P_0$ (initially),

$$P_1 = \frac{\lambda}{\mu}\,P_0 \text{ [from (23.93b)]},$$

$$P_2 = \frac{\lambda}{2\mu}\,P_1 = \frac{\lambda^2}{2\,!\,\mu^2}\,P_0 \quad \text{[putting } n = 1 \text{in (23.94b) and then substituting for } P_1],$$

$$P_3 = \frac{\lambda}{3\mu}\,P_2 = \frac{1\,\lambda^3}{3\,!\,\mu^3}\,P_0 \quad \text{[putting } n = 2 \text{n (23.94b) and then using the value of } P_2],$$

$$\dots \qquad \dots$$

In general, $$\boldsymbol{P_n = \frac{\lambda}{n\mu}\,P_{n-1} = \frac{1}{n\,!}\left(\frac{\lambda}{\mu}\right)^n P_0\,,\ \text{for } 1 \leq n \leq s}$$

$$\dots \qquad \dots$$

$$P_s = \frac{\lambda}{s\mu}\,P_{s-1} = \frac{1}{s\,!}\left(\frac{\lambda}{\mu}\right)^s P_0$$

$$P_{s+1} = \frac{\lambda}{s\mu}\,P_s = \frac{1}{s}\cdot\frac{1}{s\,!}\left(\frac{\lambda}{\mu}\right)^{s+1} P_0 \quad \textbf{[note carefully]}$$

$$P_{s+2} = \frac{\lambda}{s\mu}\,P_{s+1} = \frac{1}{s^2}\cdot\frac{1}{s\,!}\left(\frac{\lambda}{\mu}\right)^{s+2} P_0$$

Again, in general,

$$\boldsymbol{P_n = P_{s+(n-s)} = \frac{1}{s^{n-s}}\cdot\frac{1}{s\,!}\left(\frac{\lambda}{\mu}\right)^n P_0\,,\ \text{for } n \geq s.}$$

Now, find P_n using the fact $\sum_{n=0}^{\infty} P_n = 1$.

This can be broken as $\underset{(1 \le n \le s-1)}{\underset{\uparrow}{\sum_{n=0}^{s-1} P_n}} + \underset{(n \ge s)}{\underset{\uparrow}{\sum_{n=s}^{\infty} P_n}} = 1$.

or $$\sum_{n=0}^{s-1}\left[\frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n P_0\right] + \sum_{n=s}^{\infty}\left[\frac{1}{s^{n-s}} \cdot \frac{1}{s!}\left(\frac{\lambda}{\mu}\right)^n P_0\right] = 1.$$

or $$P_0\left[\sum_{n=0}^{s-1} \frac{s^n}{n!}\left(\frac{\lambda}{s\mu}\right)^n + \frac{1}{s!}\sum_{n=s}^{\infty} \frac{s^n}{s^{n-s}} \cdot \left(\frac{\lambda}{s\mu}\right)^n\right] = 1.$$

or $$P_0\left[\sum_{n=0}^{s-1} \frac{(s\rho)^n}{n!} + \frac{s^s}{s!}\sum_{n=s}^{\infty} \rho^n\right] = 1, \text{ where } \rho = \left(\frac{\lambda}{s\mu}\right)$$

or $$P_0 = \left[\sum_{n=0}^{s-1} \frac{(s\rho)^n}{n!} + \frac{s^s}{s!}\left(\frac{\rho^s}{1-\rho}\right)^*\right]^{-1} \text{ (see foot note)} \qquad ...(23.96)$$

Here, the sum $\sum_{n=s}^{\infty} \rho^n$ is meaningful only when $\rho = (\lambda/s\mu) < 1$.

Thus steady state distribution of arrivals (n) is

$$P_n = \begin{cases} \dfrac{1}{n!}\left(\dfrac{\lambda}{\mu}\right)^n P_0, \text{ if } n = 0, 1, 2, \ldots, s-1 \\ \dfrac{1}{s!} \cdot \dfrac{1}{s^{n-s}}\left(\dfrac{\lambda}{\mu}\right)^n P_0, \text{ if } n = s, s+1, s+2, \ldots \end{cases} \qquad ...(23.97)$$

Q. 1. Obtain the system of steady-state equations for the quering model (M | M | s) : (∞ | $FCFS$), and show that

$$P_n = \begin{cases} P_n (\lambda/\mu)^n/n!, \text{ if } n = 0, 1, 2, \ldots, s \\ P_0 (\lambda/\mu)^n/(s!\, s^{n-s}), \text{ if } n = s, s+1, s+2, \ldots \end{cases}$$

where

$$P_0 = \left[\sum_{n=0}^{s-1} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n + \frac{1}{s!}\left(\frac{\lambda}{\mu}\right)^s \frac{1}{1-\lambda/\mu s}\right]^{-1}$$

[Garhwal M.Sc. (Maths.) 97, (Stat.) 95, 91]

2. Explain the model M | M | s in case of first come first served basis. Give a suitable illustration.

3. Obtain the system of steady state equations and hence find the value of P (in usual notations) where (i) $n < s$ and (ii) $n \ge s$.

4. If $P_n(t)$ denotes the probability that there are n items in an M | M | 2 queue (with arrival rate λ andmean service time $1/\mu$), what are the difference differential equations for $P_n(t)$? Deduce and solve the equations governing steady state probabilities P_n, $n = 0, 1, 2, \ldots$

5. Write a note on M | M | C : (FCFS/∞/∞) queueing model. **[Delhi MA/M.Sc (OR) 92, 90]**

(c) Measures of Model IV (A)

If $n \ge s$, a queue of n customers would consist of s customers being served together with a genuine queue of $n - s$ waiting customers, hence

(i) $$L_q = \sum_{n=s}^{\infty} (n-s) P_n = \sum_{j=0}^{\infty} j P_{s+j} \qquad \text{(putting } n - s = j\text{)}$$

$$= \sum_{j=0}^{\infty} j \frac{(\lambda/\mu)^{s+j}}{s!\, s^j} P_0 \qquad \left(\text{since } P_n = \frac{(\lambda/\mu)^n}{s!\, s^{n-s}} P_0\right)$$

$^*\sum_{n=s}^{\infty} \rho^n = \rho^s + \rho^{s+1} + \rho^{s+2} \ldots \infty = \rho^s/(1-\rho)$ (sum of a G.P. of infinite terms, where $p < 1$).

$$= \sum_{j=0}^{\infty} j\left(\frac{\lambda}{\mu}\right)^s \frac{1}{s!}\cdot\left(\frac{\lambda}{\mu s}\right)^j P_0 \qquad = \left(\frac{\lambda}{\mu}\right)^s \frac{1}{s!} P_0 \sum_{j=0}^{\infty} j\rho^j, \; (\lambda/\mu s = \rho)$$

$$= \left(\frac{\lambda}{\mu}\right)^s \frac{1}{s!} P_0 \sum_{j=0}^{\infty} \rho\,(j\,\rho^{j-1}) \qquad = \left(\frac{\lambda}{\mu}\right)^s \frac{1}{s!} P_0\,\rho\left[\sum_{j=0}^{\infty} \frac{d}{d\rho}(\rho^j)\right]$$

$$= \left(\frac{\lambda}{\mu}\right)^s \frac{1}{s!} P_0\,\rho\,\frac{d}{d\rho}\left[\sum_{j=0}^{\infty} \rho^j\right] \qquad = \left(\frac{\lambda}{\mu}\right)^s \frac{1}{s!} P_0\,\rho\,\frac{d}{d\rho}[1+\rho+\rho^2+\rho^3+\ldots\infty]$$

$$= \left(\frac{\lambda}{\mu}\right)^s \frac{1}{s!} P_0\,\rho\,\frac{d}{d\rho}\left[\frac{1}{1-\rho}\right] = \left(\frac{\lambda}{\mu}\right)^s \frac{1}{s!} P_0\,\rho\,\frac{1}{(1-\rho)^2}$$

$$= P_s\cdot\frac{\rho}{(1-\rho)^2}, \text{ where, for convenience, } P_s = \frac{(\lambda/\mu)^s P_0}{s!}.$$

Therefore,

$$L_q = \frac{\rho\,(s\rho)^s}{s!\,(1-\rho)^2}\cdot P_0 = P_s\,\frac{\rho}{(1-\rho)^2} \qquad \left(\text{since } s\rho = \frac{\lambda}{\mu}\right) \qquad \ldots(23.98)$$

$$L_s = \sum_{n=0}^{\infty} n\,P_n = \sum_{n=0}^{s-1} n\,P_n + \sum_{n=s}^{\infty} n\,P_n = \sum_{n=0}^{s-1} n\,P_n + \sum_{n=s}^{\infty} (n-s)\,P_n + s\sum_{n=s}^{\infty} P_n$$

$$= \sum_{n=0}^{s-1} n\,P_n + L_q + s\left[1 - \sum_{n=0}^{s-1} P_n\right] = L_q + s + \sum_{n=0}^{s-1} (n-s)\,P_n$$

$$= L_q + s + \sum_{n=0}^{s-1} [(n-s)\,(s\rho)^n P_0/n!]$$

Using relationships of **section 23.12.-1**,

$$L_s = L_q + \frac{\lambda}{\mu} \qquad \ldots(23.99)$$

$$W_q = \frac{L_q}{\lambda} = P_s\cdot\frac{1}{s\mu\,(1-\rho)^2} \qquad \ldots(23.100)$$

$$W_s = \frac{L_s}{\lambda} = W_q + \frac{1}{\mu} \qquad \ldots(23.101)$$

The mean number of waiting individuals, who actually wait, is given by

$$(L \mid L>0) = \left[\sum_{n=s+1}^{\infty} (n-s)\,P_n \Big/ \sum_{n=s+1}^{\infty} P_n\right] = \frac{1}{(1-\rho)} \qquad \ldots(23.102)$$

The mean waiting time in queue for those who actually wait is given by

$$(W \mid W>0) = \frac{1}{1-\rho}\cdot\frac{1}{s\mu} = \frac{1}{s\mu-\lambda}. \qquad \ldots(23.103)$$

(d) Waiting time distribution. If the service discipline is *first come-first served* basis, then the waiting time distribution can be obtained, *i.e*,

$$P\,(T>t) = e^{-\mu t(1-\rho)},\; t>0.$$

This expression gives the probability of waiting for time greater than t exclusive of the service time. If T is the random variable which includes the service time, the expression is little more complex and is given by

$$\textbf{Prob. } (T>t) = e^{-\mu t}\left[1 + \frac{P_0\,(\lambda/\mu)^s}{s!\,(1-\rho)}\left(\frac{1-e^{-\mu t\,(s-1-\lambda/\mu)}}{s-1-\lambda/\mu}\right)\right].$$

(e) Probability distribution of busy period. It was observed that an arrival must wait in the queue if there are s or more customers in the system. Therefore, Prob. *(Busy period)* = Prob. $(n \geq s)$ = Prob. $(W > 0) = \sum_{n=s}^{\infty} P_n$

$$= \sum_{n=s}^{\infty} \frac{(\lambda/\mu)^n}{s\,!\,s^{n-s}} P_0 = P_0 \frac{(\lambda/\mu)^s}{s\,!} \sum_{j=0}^{\infty} \rho^j$$

$$= P_0 \frac{(\lambda/\mu)^s}{s\,!\,(1-\rho)} \text{ or } \frac{\mu\,(\lambda/\mu)^s P_0}{(s-1)\,!\,(s\mu - \lambda)}$$

Therefore, **Prob.** $(W > 0) = P_s/(1-\rho)$ **where** $P_s = \dfrac{(\lambda/\mu)^s P_0}{s\,!}$ ...(23.104)

The probability that there will be someone waiting

$$= \sum_{n=s+1}^{\infty} P_n = P_s\,\rho/(1-\rho) \qquad ...(23.105)$$

(f) Average number of idle serves $= s -$ (average number of customers served)

(g) Efficiency of $M \mid M \mid s$ **model** $= \dfrac{\textbf{(average number of customers served)}}{\textbf{(total number of customers served)}}$

Q. 1. For the case of s channels, Poisson arrivals with mean arrival rate λ and exponential service with mean service rate μ, show that the probability that an arrival has to wait is given by the formula :

$$P(n \geq s) = \frac{\mu\,(\lambda/\mu)^s P_0}{(s-1)\,!\,(s\mu - \lambda)}$$

2. Obtain the steady state solution for the model $M \mid M \mid c : (\infty \mid FIFO)$ in waiting time problem. Obtain the mean queue length, the average number of customers in the system, and the average waiting time in the system and queue, respectively. Explain why M is used in this model. **[Agra 98]**

3. (a) Explain M | M | 2 Queues in series model and derive its steady state solution. **[Garhwal M.Sc (Math.) 94, 93]**
(b) Explain M | M | 2 biseries Queue model and derive its steady state solution. **[Garhwal M.Sc. (Maths) 93]**

4. Describe (M | M | C) : (FIFO, ∞) system and state its important characteristics. **[IGNOU 2000 (June); Garhwal M.Sc. (Stat.) 92]**

5. What is a multi-channel queueing problem ? Deudce the difference-differential equations for an M | M | C : (∞ | FCFS) system. Obtain the steady state distributions for the number of units in the system and of W_q, the waiting time in the queue. **[Delhi MA/M.Sc. (OR.) 95, 93]**

23.15–1. Model IV(B). $(M \mid M \mid s) : (N \mid FCFS)$

In *model IV (A)*, if the maximum number in the system is limited to N, then as in model III,

$$\lambda_n = \begin{cases} \lambda \text{ for } 0 \leq n \leq N \\ 0 \text{ for } n > N \end{cases} \text{ and } \quad \mu_n = \begin{cases} n\mu \text{ for } 0 \leq n \leq s \\ s\mu \text{ for } s \leq n \leq N \end{cases}$$

Virtually the same relationships hold between P_n and P_0 as in *Model IV(A)* with infinite capacity. Therefore,

$$P_n = \begin{cases} (s\rho)^n P_0/n\,! & \text{for } 0 \leq n \leq s \\ s^s \rho^n P_0/s\,! & \text{for } s \leq n \leq N \\ 0 & \text{for } n > N, \end{cases}$$

where P_0 may be written as

$$P_0 = \left[\sum_{n=0}^{s-1} (s\rho)^n/n\,! + \sum_{n=s}^{N} (s\rho)^n/s\,!\,(s^{n-s}) \right]^{-1}$$

$$= \begin{cases} \left[\sum_{n=0}^{s-1} (s\rho)^n/n\,! + \dfrac{(s\rho)^s}{s\,!\,(1-\rho)} (1 - \rho^{N-s+1}) \right]^{-1}, & \rho = \dfrac{\lambda}{s\mu} \neq 1 \\ \left[\sum_{n=0}^{s-1} (s\rho)^n/n\,! + \dfrac{(s\rho)^s}{s\,!} (N - s + 1) \right]^{-1}, & \rho = \dfrac{\lambda}{s\mu} = 1 \end{cases}$$

This queueing model with limited waiting room is valuable because of its relevance to many real situations and the fact that changes may be made to its properties by adjusting the number of servers or the capacity of the waiting room. However, while Poisson arrivals are common in practice, negative exponential service times are less so, and it is the second assumption in the system $(M \mid M \mid s)$ that limits its usefulness.

Measures of Model IV(B) :

In this case the queue length is finite. The arrivals beyond a certain value are not allowed. Using the result stated earlier, the average queue length could be obtained. For simplicity, assume that the waiting room has capacity greater than the number of servers. If the arrival is not allowed to wait, it implies that there is no waiting room and no queue shall be formed. However, when the queue is permitted—

(i) $$L_q = \sum_{n=s}^{N} (n-s)\,P_n = \sum_{n=s}^{N} \frac{(n-s)\,(s\rho)^n}{s\,!\,s^{n-s}}\,P_0$$

$$= \frac{(s\rho)^s P_0}{s\,!} \sum_{x=0}^{N-s} x\rho^x = \frac{(s\rho)^s}{s\,!} P_0 \sum_{x=0}^{N-s} \rho \frac{d}{d\rho}(\rho^x)\,; \qquad (\text{setting } n-s=x)\left(\text{since } \frac{d}{d\rho}\rho^x = x\rho^{x-1}\right)$$

$$= \frac{(s\rho)^s}{s\,!} P_0\,\rho \frac{d}{d\rho}\left[\sum_{x=0}^{N-s} \rho^x\right]^* \quad \textbf{(see foot-note)}$$

$$= \frac{(s\rho)^s}{s\,!} P_0\,\rho \frac{d}{d\rho}\left[\frac{1-\rho^{N-s+1}}{1-\rho}\right]$$

$$= \frac{(s\rho)^s P_0\,\rho}{s\,!\,(1-\rho)^2}\left[(1-\rho^{N-s+1}) - (1-\rho)\,(N-s+1)\,\rho^{N-s}\right] \qquad (\text{since } L_q \text{ cannot be} - \text{ive})$$

(ii) Average number in the service facility

$$= L_s - L_q = \sum_{n=0}^{N} n\,P_n - \sum_{n=s}^{N} (n-s)\,P_n = \sum_{n=0}^{s-1} n\,P_n + \sum_{n=s}^{N} n\,P_n - \sum_{n=s}^{N} (n-s)\,P_n$$

$$= \sum_{n=0}^{s-1} n\,P_n + s\sum_{n=s}^{N} P_n = \sum_{n=0}^{s-1} n\,P_n + s\left[\sum_{n=0}^{N} P_n - \sum_{n=0}^{s-1} P_n\right]$$

$$= \sum_{n=0}^{s-1} nP_n + s\left[1 - \sum_{n=0}^{s-1} P_n\right] = s + \sum_{n=0}^{s-1} (n-s)\,P_n = s + P_0 \sum_{n=0}^{s-1} [(n-s)\,(\rho s)^n.$$

(iii) To obtain the average waiting time in the system, (use the Littles formula)

$W_s = L_s/\lambda'$, where $\lambda' = \lambda\,(1 - P_N)$

(iv) $W_q = W_s - (1/\mu) = L_q/\lambda'$.

23.15–2. Model IV (C). $(M \mid M\, s)$: $(s \mid FCFS)$

This model is same as ***Model IV (B)*** except that $N = s$ here. In this situation, no waiting queue is allowed to form.

The value of P_n can be obtained by simply substituting $N = s$ in the result of Model IV (B), and this is called the *Erlang's first formula.* Thus,

$$P_n = \begin{cases} P_0\,(s\rho)^n/n\,!\,, & 0 \le n \le s \\ 0 & , \text{otherwise} \end{cases}$$

where $P_0 = \left[\sum_{n=0}^{s} (s\rho)^n/n\,!\right]^{-1}$ which is known as *Erlang's loss formula.*

23.15–3. Model IV (D) : Single Service Counter and Arrivals Through Multiple Channels

Let us consider a single service station having service rate μ and s number of channels with arrival rate λ for each channel. Thus

---

* $\sum_{x=0}^{N-s} \rho^x = 1 + \rho + \rho^2 + \ldots + \rho^{N-s}$ [G.P. of $N-s+1$ terms) $= \frac{1-\rho^{N-s+1}}{1-\rho}$]

P_n = Prob. that there will be n (less than s) units in the system

$$= P_0 \left(\frac{\lambda}{\mu}\right)^n \frac{s\,!}{(s-n)\,!}$$

To find the probability of an empty system, *i.e.* P_0, we have

$$P_0 + P_1 + P_2 + \ldots + \infty = 1 \quad \text{or} \quad P_0 + P_0\left(\frac{\lambda}{\mu}\right)\frac{s\,!}{(s-1)\,!} + P_0\left(\frac{\lambda}{\mu}\right)^2 \frac{s\,!}{(s-2)\,!} + \ldots \infty = 1$$

or $$P_0\left[1 + \left(\frac{\lambda}{\mu}\right)\frac{s\,!}{(s-1)\,!} + \left(\frac{\lambda}{\mu}\right)^2 \frac{s\,!}{(s-2)\,!} + \ldots \infty\right] = 1$$

or $$P_0 = \left[1 + \left(\frac{\lambda}{\mu}\right)\frac{s\,!}{(s-1)\,!} + \left(\frac{\lambda}{\mu}\right)^2 \frac{s\,!}{(s-2)\,!} + \ldots \infty\right]^{-1}$$

The expected number of customers in the system

$$= 1.P_1 + 2.P_2 + 3.P_3 + \ldots = P_0\left(\frac{\lambda}{\mu}\right)\frac{s\,!}{(s-1)\,!} + 2\,P_0\left(\frac{\lambda}{\mu}\right)^2 \frac{s\,!}{(s-2)\,!} + \ldots$$

The following example will make this situation clear.

Example 30. *In machine maintenance, a mechanic repairs four machines. The meantime between service requirement is 5 hours for each machine and forms an exponential distribution. The mean repair time is one hour and also follows the same distribution pattern. Machine down time costs Rs. 25 per hour and the machine costs Rs. 55 per day of 8 hrs.*

(i) Find the expected number of operating machines.

(ii) Determine the expected down time cost per day.

(iii) Would it be economical to engage two mechanics each repairing only two machines ?

Solution. In this example, $s = 4$, $\lambda = 1/5$ per hour and $\mu = 1$ hour.

$$\therefore \quad P_0 = \left[1 + \left(\frac{1}{5}\right)\frac{4\,!}{3\,!} + \left(\frac{1}{5}\right)^2 \frac{4\,!}{2\,!} + \left(\frac{1}{5}\right)^3 \frac{4\,!}{1\,!} + \left(\frac{1}{5}\right)^4 \frac{4\,!}{0\,!}\right]^{-1}$$

$$= [1 + 0.8 + 0.48 + 0.192 + 0.0384]^{-1} = 0.4$$

Expected number of machines in the system

$$= 1P_1 + 2P_2 + 3P_3 + 4P_4 = P_0\,(1 \times 0.80 + 2 \times 0.48 + 3 \times 0.192 + 4 \times 0.0384)$$

$$= 0.4 \times 2.4896 = 0.996 \cong 1.$$

Hence down time per day is 8 machine hours.

(i) The number of operating machines in the system

$$= 4 - \text{Expected number of machines in the system}$$

$$= 4 - 1 = 3.$$

(ii) Cost of expected down time per day

$$= 1 \times 8 \times 25 = \text{Rs. } 200.$$

$\therefore$ Total cost including operator = 200 + 55 = Rs. 255.

(iii) In case 4, machines are distributed to 2 mechanics each to attend 2 machines,

$$P_0 = \left[1 + \left(\frac{\lambda}{\mu}\right)\frac{2\,!}{1\,!} + \left(\frac{\lambda}{\mu}\right)^2 \frac{2\,!}{0\,!}\right]^{-1}$$

$$= [1 + 0.40 + 0.08]^{-1} = 1/1.48 = 0.68.$$

Expected number of machines in the system = $1P_1 + 2P_2 = P_0\,(0.80 + 2 \times 0.48) = 1.70$

Total downtime cost per day for 2 sets of machines to each operator

$$= 0.38 \times 2 \times 8 \times 25 = \text{Rs. } 152.$$

Total cost including 2 operators per day = 152 + 2 × 55 = Rs. 262.

Thus it would not be economical to engage two mechanics.

Q. 1. Describe the general problem of $M\,|\,M\,|\,k$ queueing system and deduce an explicit expression for the steady-state probability of the length of the queue in an $M\,|\,M\,|\,1$ system.

2. State the basic axioms governing Poisson queues. Find the distribution of arrivals for the Poisson queues.

23.15–4. Illustrative Examples on Model IV

Example 31. *A supermarket has two girls ringing up sales at the counters. If the service time for each customer is exponential with mean 4 minutes, and if people arrive in a Poisson fashion at the counter at the rate of 10 per hour.* **[Meerut 99, 98; Banasthali (Maths.) 93]**

(*a*) *Calculate the probability that an arrival will have to wait for service.*

(*b*) *Find the expected percentage of idle time for each girl ?*

(*c*) *If a customer has to wait, find the expected length of his waiting time.*

[JNTU (B. Tech.) 2000; Meerut (Maths) 91]

Solution. (*a*) Probability of having to wait for service is

$$P(W > 0) = \frac{(\lambda/\mu)^s}{s\,!\,(1-\rho)} P_0 \quad \text{[from eqn. (23.104)]}$$

Here, $\lambda = 1/6,\ \mu = 1/4,\ s = 2,\ \rho = \lambda/\mu s = 1/3.$

Now, compute $P_0 = \left[\sum_{n=0}^{s-1} \frac{(s\rho)^n}{n\,!} + \frac{(s\rho)^s}{s\,!\,(1-\rho)}\right]^{-1}$ [from eqn. (23.96)]

$$= \left[\sum_{n=0}^{1} \frac{(2\times \frac{1}{3})^n}{n\,!} + \frac{(2.\frac{1}{3})^2}{2\,!\,(1-\frac{1}{3})}\right]^{-1} = \left[1 + 2/3 + \frac{4/9}{2\times 2/3}\right]^{-1} = 1/2.$$

Thus, $$\text{Prob.}\ (W > 0) = \frac{(4/6)^2 . \frac{1}{2}}{2\,!\,(1-\frac{1}{3})} = 1/6.$$ **Ans.**

(b) The fraction of the time the service remains busy (*i.e.* traffic intensity) is given $\rho = \lambda/s\mu = 1/3$.
Therefore, the fraction of the time the service remains idle is

$$= (1 - \tfrac{1}{3}) = \tfrac{2}{3} = 67\% \text{ (nearly)}.$$

(c) $(W \mid W > 0) = \frac{1}{1-\rho}\cdot\frac{1}{s\mu} = \frac{1}{1-\frac{1}{3}}\cdot\frac{1}{2\times \frac{1}{4}} = 3$ minutes. **Ans.**

Example 32. *A telephone exchange has two long distance operators. The telephone company finds that, during the peak load, long distance calls arrive in a Poisson fashion at an average rate of 15 per hour. The length of service on these calls is approximately exponentially distributed with mean length 5 minutes.*

[Meerut (Maths.) 93P]

(*a*) *What is the probability that a subscriber will have to wait for his long distance call during the peak hours of the day ?*

(*b*) *If the subscribers will wait and serviced in turn, what is the expected waiting time ?*

Establish the formula used. **[Rohilkhand 94; Garhwal M.Sc (Stat) 91]**

Solution. Here $s = 2,\ \lambda = 15/60 = 1/4,\ \mu = 1/5$.

Therefore, first compute

$$P_0 = \left[\sum_{n=0}^{s-1} \frac{(s\rho)^n}{n\,!} + \frac{(s\rho)^s}{s\,!\,(1-\rho)}\right]^{-1} = \left[\sum_{n=0}^{1} \frac{(5/4)^n}{n\,!} + \frac{(5/4)^2}{2\,!\,(1-5/8)}\right]^{-1}$$

$$= \frac{1}{1 + \frac{5}{4} + (\frac{5}{4})^2 . \frac{1}{2} . \frac{8}{3}} = 3/13\,.$$

(*a*) $\text{Prob.}\ (W > 0) = \frac{(\lambda/\mu)^s}{s\,!\,(1-\rho)} P_0 \quad = \frac{(5/4)^2 . 3/13}{2\,!\,(1-5/8)} = 25/52 \simeq 0.48.$ **Ans.**

(*b*) $W_q = L_q/\lambda = \frac{1}{\lambda}\cdot\frac{\rho\,(s\rho)^s}{s\,!\,(1-\rho)^2} P_0,$

$$= 4\cdot\frac{5/8\,(5/4)^2}{2\,!\,(1-5/8)^2}\cdot\frac{3}{13} = \frac{125}{39} \simeq 3.2 \text{ minutes.}$$ **Ans.**

Example 33. *A bank has two tellers working on savings accounts. The first teller handles withdrawals only. The second teller handles deposits only. It has been found that the service time distribution for the deposits and withdrawals both are exponential with mean service time 3 minutes per customer. Depositors are found to arrive in a Poisson fashion throughout the day with mean arrival rate 16 per hour. Withdrawers also arrive in a Poisson fashion with mean arrival rate 14 per hour. What would be the effect on the average waiting time for depositors and withdrawers if each teller could handle both withdrawals and deposits. What would be the effect if this could only be accomplished by increasing the service time to 3.5 minutes ?*

[IAS (Main) 98; Agra 98]

Solution. In the first case, $\lambda_1 = 16/60 = 4/15$, $\mu = 1/3$.

(a) $W_q^{(1)} = \dfrac{\lambda_1}{\mu(\mu - \lambda_1)} = \dfrac{4}{15 \times 1/3\,(1/3 - 4/15)} = 12$ min. **Ans.**

(b) In the second case, $\lambda_2 = 14/60 = 7/30$, $\mu = 1/3$.

$$W_q^{(2)} = \frac{\lambda_2}{\mu(\mu - \lambda_2)} = \frac{7/30}{1/3\,(1/3 - 7/30)} = 7 \text{ minutes.}$$ **Ans.**

(c) Now combining these two cases,

$$\mu = \frac{1}{3}, \text{ but } \lambda = \frac{14+16}{60} = \frac{1}{2}.$$

Since this becomes the case of two service stations, *i.e.*, $s = 2$, then

$$\rho = \lambda/s\mu = 3/4.$$

Therefore,

$$P_0 = \left[\sum_{n=0}^{s-1} \frac{(s\rho)^n}{n!} + \frac{(s\rho)^s}{s!\,(1-\rho)}\right]^{-1}$$

$$= \left[\sum_{n=0}^{1} \frac{(2\,.\,3/4)^n}{n!} + \frac{(2\,.\,3/4)^2}{(2!)\,(1-3/4)}\right]^{-1} = \frac{1}{1 + 3/2 + 9/2} = \frac{1}{7}.$$

$$W_q = \frac{1}{\lambda} \cdot \frac{\rho\,(s\rho)^s}{s!\,(1-\rho)^2} P_0,\ = 2 \cdot \frac{3/4\,.\,(3/2)^2}{(2!)\,.\,(1-3/4)^2} \times \frac{1}{7} = \frac{27}{7} = 3.86 \text{ minutes.}$$ **Ans.**

(d) If $\mu = 10/35 = 2/7$, $\lambda = 1/2$, then $\rho = \lambda/s\mu = 7/8$. Therefore,

$$P_0 = \left[\sum_{n=0}^{1} \frac{(7/4)^n}{n!} + \frac{(7/4)^2}{(2!)\,(1-7/8)}\right]^{-1} = \frac{1}{1 + 7/4 + 49/4} = \frac{1}{15}.$$

$$\therefore \quad W_q = \frac{1}{\lambda} L_q = \frac{1}{\lambda} \cdot \frac{\rho\,(s\rho)^s}{s!\,(1-\rho)^2} P_0 = 2 \cdot \frac{7/8\,.\,(7/4)^2}{(2!)\,.\,(1-7/8)^2} \cdot \frac{1}{15} = \frac{343}{30} = 11.4 \text{ minutes.}$$ **Ans.**

Example 34. *Four counters are being run on the frontier of a country to check the passports and necessary papers of the tourists. The tourists choose a counter at random. If the arrival at the frontier is Poisson at the rate λ and the service time is exponential with parameter $\lambda/2$ What is the steady average queue at each counter ?*

Solution. Here $s = 4$, $\mu = \lambda/2$, $\rho = \lambda/\mu s = 1/2$. Therefore,

$$P_0 = \left[\sum_{n=0}^{3} \frac{2^n}{n!} + \frac{4^4}{4!} \cdot \frac{(1/2)^4}{(1-1/2)}\right]^{-1} = \left[\frac{2^0}{0!} + \frac{2^1}{1!} + \frac{2^2}{2!} + \frac{2^3}{3!} + \frac{256}{24} \times \frac{1}{8}\right]^{-1}$$

$$= \left[1 + 2 + 2 + \frac{8}{6} + \frac{4}{3}\right]^{-1}$$

$$= \left[5 + \frac{8}{3}\right]^{-1} = \frac{3}{23}.$$

But, expected queue length $(L_q) = \dfrac{(\lambda/\mu)^s}{s} \cdot \dfrac{\rho}{(1-\rho)^2} P_0 = \dfrac{2^4}{4!} \cdot \dfrac{1/2}{(1/2)^2} \cdot \dfrac{3}{23} = \dfrac{4}{23}.$ **Ans.**

Example 35. *A company currently has two toolcribs, each having a single clerk, in its manufacturing area. One toolcrib handles only the tools for the heavy meachinery, while the second one handles all other tools. It is observed that for each toolcrib the arrivals follow a Poisson distribution with a mean of 20 per hour, and the service time distribution is negative exponential with a mean of 2 minutes.*

The tool manager feels that, if toolcribs are combined in such a way that either clerk can handle any kind of tool as demand arises, would be more efficient and the waiting problem could be reduced to some extent. It is believed that the mean arrival rate at the two toolcribs will be 40 per hour; while the service time will remain unchanged.

Compare in status of queue and the proposal with respect to the total expected number of machines at the toolcrib(s), the expected waiting time including service time for each mechanic and probability that he has to wait for more than five minutes.

Solution. ***When toolcrib works independently :***

$$\lambda = 20 \text{ per hr.}, \ \mu = 60/2 = 30 \text{ per hr.}, \ \rho = \lambda/\mu = 20/30 = 2/3$$

$$L_s = \frac{\rho}{1-\rho} = \frac{2/3}{1-2/3} = 2.$$

Therefore, on an average 2-persons are waiting in each toolcrib, *i.e.*,

$$W_s = L_s/\lambda = 2 \times 3 = 6 \text{ minutes (each)} \qquad \{\lambda = 20 \text{ per hour} = \tfrac{1}{3} \text{ per minute.}\}$$

$$\text{Prob. } (T > 5) = e^{-\mu(1-\rho)t} = e^{-30(1-2/3)\times 5/60} = e^{-1/2 \times 1/3 \times 5} = 0.435.$$

When both the toolcribs work jointly :

This forms a 2-server system, *i.e.*, $\lambda = 40$, $\mu = 30$, $s = 2$, $\rho = \lambda/s\mu = 2/3$

$$P_0 = \left[\sum_{n=0}^{s-1} \frac{(\lambda/\mu)^n}{n!} + \frac{(\lambda/\mu)^s}{s!\,(1-\lambda/\mu s)}\right]^{-1} = \left[1 + \frac{\lambda}{\mu} + \frac{1}{2}\frac{(\lambda/\mu)^2}{1-\rho}\right]^{-1}$$

$$= \left[1 + \frac{4}{3} + \frac{1}{2}\left(\frac{4}{3}\right) \times \frac{1}{1-2/3}\right]^{-1} = \frac{1}{5}$$

$$L_s = L_q + \lambda/\mu$$

$$= \left(\frac{\lambda}{\mu}\right)^s \frac{\rho P_0}{s!\,(1-\rho)^2} + \frac{4}{3} = \left(\frac{4}{3}\right)^2 \times \frac{2}{3} \times \frac{1}{2 \times (1/3)^2} \times 1/5 + \frac{4}{3}\,\frac{16}{9} \times \frac{3}{5} + \frac{4}{3} = 2.4 \text{ persons}$$

$$W_s = L_s/\lambda = (2.4 \times 60)/40 = 2.4 \times 1.5 = 3.6 \text{ minutes}$$

$$P(T > 5) = e^{-\mu t}\left\{1 + \frac{P_0\,(\lambda/\mu)^s}{s!\,(1-\rho)}\left(\frac{1 - e^{-\mu t\,(s-1-\lambda/\mu)}}{s-1-\lambda/\mu}\right)\right\}$$

$$= e^{-5/2}\left[1 + \frac{1/5 \times 16/9}{2 \times 1/3}\left(\frac{1 - e^{-5/2\,(1-4/3)}}{1-4/3}\right)\right] = 0.25.$$

Now it is clear that combining the two toolcribs and asking both servers to attend the customers leads to a more efficient handling of the system. When toolcribs work independently, certain advantages are occurred but by combining the two, the net waiting time is reduced. In the former system, it is possible that one toolcrib is idle, whereas in the latter case crib may have a long queue.

Consider the case of a manufacturing plant in which materials are to be moved from one location to another. For the sake of argument, assume that the material handling equipment is of *'similar kind'*. Now, problem is :

(i) should there be a centralized pool for these equipments and all departments place their request for the equipment to the sections, or

(ii) each department will maintain a fleet of the equipment to serve their material handling heads.

It is obvious that a centralized pool or fleet will be economical however, this conclusion is based on the assumption that the location of the centralized fleet is most convenient to all departments and there are no delay or poor service because of travel time involved, paper work in making the request, and some other priority and management problems are irrelevant, keeping in mind the salient differences one could easily find which one is the most economical choice.

EXAMINATION PROBLEMS ON MODEL IV

1. A petrol pump station has two pumps. The service time follows the exponential distribution with a mean of 4 minutes and cars arrive for service in a Poisson process at the rate of ten cars per hour. Find the probability that a customer has to wait for service. What proportion of time the pumps remain idle ?
[**Ans.** 0.167, 67% for each pump]
2. A two-channel waiting line with Poisson arrival has a mean arrival rate of 50 per hour and exponential service with a mean service rate of 75 per hour for each channel. Find :
(i) The probability of an empty system. (ii) The probability that an arrival in the system will have to wait.
[**Ans.** (i) 0.83, (ii) 0.167]
3. There are two clerks in a University to receive dues from the students. If the service time for each student is exponential with mean 4 minutes, and the boys arrive in a Poisson fashion at the counter at the rate of 10 per hour.
(i) What is the probability of having to wait for service ?
(ii) What is the expected percentage of idle time for each clerk ?
[**Ans.** (i) 0.167, (ii) 0.67]
4. For a queueing system with k service stations, each having exponential service time distribution with mean service rate μ feed by a queue built up of arrival rate λ, find :
(i) average number of customers in the system, (ii) average waiting time of a customer in the system.
If $k = 2$ and $\mu = 5$ per minute and $\lambda = 8$ per minute
(a) What is the probability of a delay ? (b) What is the probability that at least one of the servicing station is idle ?
[**Ans.** (i) 4.44, (ii) 0.56 minutes]
5. A toolcrib is operated by M servers, and demands for service arrives randomly (Poisson arrival) at mean rate of 14 per minute. The mean service time per server is 1.25 minutes (exponentially distributed). Finite population effects may be ignored. If the average hourly pay rate of a tool crib operator is Rs. 2 per hour and the average hourly pay rate of production employee is Rs. 4 per hour, determine the optimum value of M. (Assume 8 hour working day)
6. A railway goods traffic section has 4 claims Assistants. Customers with claims against the railway are observed to arrive in a Poisson fashion at an average rate of 24 per 8 hour day for 6 days. The amount of time the claims Assistant spends with the claimant is found to have an exponential distribution with a mean service time of 40 min. Service is given in the order of appearance of the customers.
(i) How many hours/week can a claims Assistant expect to serve the claimant ?
(ii) How much time does a claimant on the average spend in the goods traffic office.
[**Ans.** (i) 72 hours, (ii) 47.2 minutes]
7. In a huge workshop tools are stored in a tool crib. Mechanics arrive at the tool crib for taking the tools and hand them back after they have used them. It is found that the average time between arrivals of mechanics at the crib is 35 seconds. A clerk at the crib has been found to take on an average 50 seconds to serve a mechanic (either hand him the tools if he requests them or receive tools if he is returning them). If the labour cost of a clerk is Re. 1 per hour and that of a mechanic is Rs. 2.5 per hour, find out how many clerks should be appointed at the tool crib to minimize the total cost of mechanics' waiting time plus the clerks idle time.
[**Ans.** Two clerks should be appointed.]
8. A general insurance company has three claims adjusters in its branch office. People with claims against the company are found to arrive in Poisson fashion, at an average rate of 20 per-8-hour day. The amount of time that an adjuster spends with a claimant is found to have a negative exponential distribution, with mean service time 40 minutes. Claimants are processed in the order of their appearance.
(a) How many hours a week can an adjuster expect to spend with claimants ?
(b) How much time, on an average, does a claimant spend in the branch office. **[Rohilkhand 93]**
[**Ans.** (a) 22.2 hours, (b) 49 minutes.]
9. Given an average arrival rate of 20 per hour, is it better for a customer to get service at a single channel with mean service rate of 2 customers or at one of two channels in parallel with mean service rate of 11 customers for each of the two channels ? Assume that both queues are of $M \mid M \mid s$ types.
[**Ans.** It is better to get service at two channels in parallel.]
10. Find the probability that there are no customers in the system given that :
(a) Number of channels in parallel = 3 (b) Mean arrival rate = 24/hour
(c) Mean service rate in each channel = 10/hour
Compare the average time that a customer is in the system for the following 2 systems :
(i) 5 channels in parallel with a mean service rate of 10/hour.
(ii) 1 channel with a mean service rate of 30/hour.
[**Ans.** $P_0 = 0.019$, (i) 6 minutes, (ii) 10 minutes]
11. A tax consulting firm has four service stations (counters) in its office to receive people who have problems and complaints about their income, wealth and sales taxes. Arrivals average 80 persons in an 8-hour service day. Each tax adviser spends an irregular amount of time in servicing the arrivals which have been found to have an exponential distribution. The average service time is 20 minutes. Calculate the average number of customers in the system, average number of customers waiting for service, average time a customer spends on system, and average waiting time for a

customer. Calculate how many hours each week does a tax adviser spend in performing his job. What is the probability that a customer has to wait before he gets service ? What is the expected number of ideal tax advisers at any specified time ? **[IAS (Main) 96]**

[**Hint.** Here $\lambda = 10$/hour, $\mu = 3$/hour, $s = 4$ and $P_0 = 0.0213$.

(i) $L_s = 6.61$, (ii) $L_q = L_s - \lambda/\mu = 3.28$ customers, (iii) $W_s = 0.66$ hour, (iv) $W_q = W_s - (1/\mu) = 0.33$ hour.

(v) $P = \lambda/s\mu = 0.8333$, so the expected time spent in servicing during 8-hour day is $8 \times 0.833 = 6.66$ hours. Thus, the tax adviser is busy 33.3 hours based on a 30 hour week, (vi) $P(n \geq 0) = 0.65$,

(vii) Since $-P_n = \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n P_0$, the expected number of idle advisers is $= 4P_0 + 3P_1 + 2P_2 + 1P_3 = 0.666$]

12. A barber shop has two barbers and three chairs for waiting customers. Assume that customers arrive in a Poisson fashion at a rate of 5 per hour and that each barber services customers according to an exponential distribution with mean of 15 minutes. Further, if a customer arrives and there are no empty chairs in the shop he will leave. Find the steady state probabilities. What is the probability that the shop is empty ? What is the expected number of customers in the shop.

[**Ans.** $P_n = \frac{1}{n!}(0.28)(1.2)^n$, $0 \leq n < 2$ $= (0.6)^n (0.28)$, $2 \leq n \leq 3$, $P_0 = 0.28$, $L_s = 3$ customers approx.]

13. A car servicing station has two bays where service can be offered simultaneously. Due to space limitation, only four cars are accepted for servicing. The arrival pattern is Poisson with 12 cars per day. The service time in both the bays is exponentially distributed with $\mu = 8$ cars per day per bay. Find the average number of cars in the service station, the average number of cars waiting to be serviced and the average time a car spends in the system. **[I.A.S. (Maths) 89]**

14. In a factory cafeteria the customers have to pass through three counters. The customers buy coupons at the first counter, select and collect the snacks at the second counter and collect tea at the third. The server at each counter takes on an average 1.5 minutes though the distribution of service time is approximately expontential. It arrival of customers to the cafeteria is approximately Poisson at an average rate of 6 per hour, calculate :
(i) the average time a customer spends waiting in the cafeteria,
(ii) the average time of getting the service, and
(iii) the most profitable time of getting the service. **[Virbhadrah 2000]**

15. In Indian coffee Cafe centre, it was observed that there is only one bearer who fakes exactly 4 minutes to serve a cup of coffee once the order has been placed with him. If the students arrive is the cafe centre at an average rate of 10 per hour, how much time one has expected to spend waiting for his turn to place the order. **[JNTU (Mech. Prod.) May 2004]**

23.16. NON-POISSON QUEUEING MODEL

The *queues* in which arrivals and/or departures may not follow the *Poisson* axioms are called *Non-Poisson* queues. The development of these queueing systems becomes more difficult, mainly because the Poisson axioms no longer hold good. However, following techniques are usually adopted for the development of non-Poisson queues :

(a) ***Phase Technique.*** This technique is used when an arrival demands phases of service, say k in number.

(b) ***Imbedded Markov Chain Technique.*** The technique by which non-Markovian queues are reduced to *Markovian* is termed as *Imbedded Markov Chain Technique.*

(c) ***Supplementary Variable Technique.*** When one or more random variables are added to convert a non-Markovian process into a Markovian one, the technique involved is called *Supplementary Variable Technique.* This technique is used for the queueing models : $GI \mid G \mid C$, $M \mid G \mid 1$, $GI \mid M \mid S$, $GI \mid E_k \mid 1$, $D \mid E_k \mid 1$. But we have not presented these models here initially.

However we shall introduce the queueing system $M \mid E_k \mid 1$ and the steady state results of $M \mid G \mid 1$.

23.16–1. Model V(A). ($M \mid E_k \mid 1$) : ($\infty \mid FCFS$)

In this queueing system of *Poisson* arivals, *Erlang* service time with k phases and single server, suppose

$P_n(t)$ = probability that there are n phases in the system (waiting and in service) at time t,

n = total number of phases in the system (waiting and in service), and

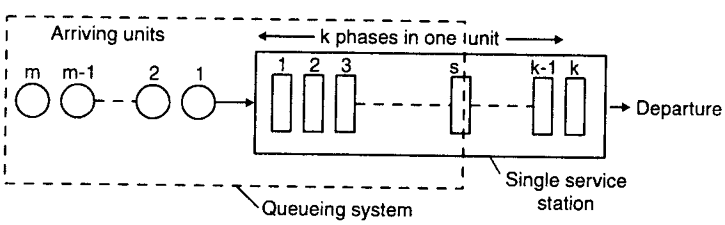


Fig. 23.20

k = number of phases in one unit.

Consider that one unit is served in k phases one by one. So each arrival will increase the number of phases by k in the system, and each time a phase is completed, the number of phases in the system is decreased by unity. Thus, if there are m units waiting in the queue and one unit is already running under service at the sth phase, then $n = mk + s$.

If μ is the number of units served per unit time, then μk will be the number of phases served per unit time, *i.e.*, $1/\mu k$ will be the time taken by one service at each phase. Therefore,

$\lambda_n = \lambda$ phases arrive per unit time, and $\mu_n = \mu k$ phases served per unit time.

(a) To obtain the system of steady state equations.

Use the same procedure as discussed for earlier models to obtain

$$P_0(t+\Delta t) = P_0(t)[1-\lambda\Delta t] + P_1(t)\,k\mu\Delta t + O(\Delta t),\ n = 0, \qquad ...(23.106)$$

$$P_n(t+\Delta t) = P_n(t)[1-(\lambda+k\mu)\Delta t] + P_{n-k}(t)\,.\,\lambda\Delta t + P_{n+1}(t)\,k\mu\,\Delta t + O(\Delta t) \qquad ...(23.107)$$

where $n \geq 1$ and $P_j(t) = 0$ for $j < 0$,

Now transform the equations (23.106) and (23.107) into the system of differential equations given by

$$P_0'(t) = -\lambda P_0(t) + k\mu P_1(t),\ n = 0 \qquad ...(23.106a)$$

$$P_n'(t) = -(\lambda+k\mu)P_n(t) + \lambda P_{n-k}(t) + k\mu P_{n+1}(t),\ n \geq 1. \qquad ...(23.107a)$$

The system of equations (23.106a) and (23.107a) gives rise to the following steady state system of difference equations

$$0 = -\lambda P_0 + k\mu P_1,\ n = 0 \qquad ...(23.106b)$$

$$0 = -(\lambda+k\mu)P_n + \lambda P_{n-k} + k\mu P_{n+1},\ n \geq 1. \qquad ...(23.107b)$$

Using the notation $\rho = \lambda/k\mu$ for the traffic intensity of the phases, the system of eqns. (23.106b) and (23.107b) becomes

$$P_1 = \rho P_0,\ n = 0 \qquad ...(23.106c)$$

$$(1+\rho)P_n = P_{n+1} + \rho P_{n-k},\ n \geq 1 \ \text{[dividing (23.107b) by } \mu k \text{ and putting } \lambda/\mu k = \rho] \qquad ...(23.107c)$$

which is the required system of difference equations.

(b) To solve the system of equations by the method of Generating Functions (G.F.)

By the definition of G.F.

$$\text{G.F.} \equiv P(z) = \sum_{n=0}^{\infty} P_n z^n \qquad ...(23.108)$$

Now, multiplying the equation (23.107c) by z^n and taking summation from 1 to ∞ (this equation is true only when $n \geq 1$)

$$(1+\rho)\sum_{n=1}^{\infty} P_n z^n = \sum_{n=1}^{\infty} P_{n+1} z^n + \rho \sum_{n=1}^{\infty} P_{n-k} z^n \qquad ...(23.109)$$

Adding ρP_0 on the left side and P_1 on the right side of equation (23.109) [since $P_1 = \rho P_0$ from equation (23.106c)], then

$$\rho P_0 + (1+\rho)\sum_{n=1}^{\infty} P_n z^n = (P_1 + \sum_{n=1}^{\infty} P_{n+1} z^n) + \rho \sum_{n=1}^{\infty} P_{n-k} z^n$$

or
$$(1+\rho)[P_0 + \sum_{n=1}^{\infty} P_n z^n] - P_0 = \sum_{n=0}^{\infty} P_{n+1} z^n + \rho \sum_{n=1}^{\infty} P_{n-k} z^n$$

or
$$(1+\rho)\sum_{n=0}^{\infty} P_n z^n - P_0 = (1/z)\sum_{n=0}^{\infty} P_{n+1} z^{n+1} + \rho \sum_{n=k}^{\infty} P_{n-k} z^n$$

[since $P_{n-k} = 0$ for all $n-k < 0$, *i.e.*, for $n < k$]

Substituting $n + 1 = i$ in Ist term and $n - k = m$ in IInd term or right side, and adjusting the corresponding limits

$$(1+\rho)\sum_{n=0}^{\infty} P_n z^n - P_0 = (1/z)\sum_{i=1}^{\infty} P_i z^i + \rho \sum_{m=0}^{\infty} P_m z^{k+m}$$

or $$(1+\rho)\sum_{n=0}^{\infty} P_n z^n - P_0 = (1/z)\left[\sum_{i=0}^{\infty} P_i z^i - P_0\right] + \rho z^k \sum_{m=0}^{\infty} P_m z^m$$

or $$(1+\rho)\,P(z) - P_0 = (1/z)\,[P(z) - P_0] + \rho z^k P(z) \left[\text{since } P(z) = \sum_{n=0}^{\infty} P_n z^n\right]$$

or $$P(z)\,[(1/z) + \rho z^k - (1+\rho)] = P_0\,[(1/z - 1)]$$

or $$P(z) = \frac{P_0(1-z)}{1+\rho z^{k+1} - (1+\rho)z} = \frac{P_0(1-z)}{(1-z) - \rho z(1-z^k)}$$

or $$P(z) = \frac{P_0}{1-\rho z\left(\dfrac{1-z^k}{1-z}\right)} = P_0\left[1-\rho z\left(\frac{1-z^k}{1-z}\right)\right]^{-1}$$

Using binomial theorem

$$P(z) = P_0\left[1+\rho z\left(\frac{1-z^k}{1-z}\right) + (\rho z)^2\left(\frac{1-z^k}{1-z}\right)^2 + \ldots\right] \qquad \ldots(23.110)$$

or $$P(z) = P_0 \sum_{n=0}^{\infty} (\rho z)^n \left(\frac{1-z^k}{1-z}\right)^n$$

Because it can be shown that the sum of the series $(1+z+z^2+\ldots+z^{k-1})$ is equal to $(1-z^k)/(1-z)$, it follows that

$$P(z) = P_0 \sum_{n=0}^{\infty} (\rho z)^n\,[1+z+z^2+\ldots+z^{k-1}]^n \qquad \ldots(23.111)$$

Now find P_0 and P_n from equation (23.111).

To find P_0 : On putting $z = 1$ in equn. (23.111),

$$P(1) = P_0 \sum_{n=0}^{\infty} \rho^n\,[1+1+1+\ldots k \text{ times}]^n = P_0 \sum_{n=0}^{\infty} \rho^n k^n$$

or $$P(1) = P_0/(1-\rho k) \quad \left[\text{since} \sum_{n=0}^{\infty} (\rho k)^n = 1/(1-\rho k) \text{ for sum of infinite GP}\right]$$

On the other hand, putting $z = 1$ in equation (23.108),

$$P(1) = \sum_{n=0}^{\infty} P_n = 1 \text{ (by the law of total probability} \sum_{n=0}^{\infty} P = 1) \qquad \ldots(23.113)$$

Comparing the equations (23.112) and (23.113),

$$P_0/(1-\rho k) = 1$$

which gives $$\mathbf{P_0 = 1 - \rho k \text{ (Imp. Note)}} \qquad \ldots(23.114)$$

To find the value of P_n :

Substituting the value of P_0 from equation (23.114) and letting $n = m$ in equation (23.110),

$$P(z) = (1-\rho k) \sum_{m=0}^{\infty} (\rho z)^m (1-z^k)^m (1-z)^{-m} \qquad \ldots(23.115)$$

Now $$(1-z^k)^m = 1 - {}^m c_1 (z^k) + {}^m c_2 (z^k)^2 + \ldots + (-1)^m\, {}^m c_m (z^k)^m$$

$$= \sum_{i=0}^{m} (-1)^i \binom{m}{i} z^{ik}$$

and $$(1-z)^{-m} = \sum_{j=0}^{\infty} (-1)^j \binom{-m}{j} z^j = \sum_{j=0}^{\infty} (-1)^{2j} \binom{m+j-1}{j} z^j \quad \left[\text{since} \binom{-m}{j} = (-1)^j \binom{m+j-1}{j}\right]$$

Hence the equation (23.115) becomes

$$P(z) = (1-\rho k) \sum_{m=0}^{\infty} (\rho z)^m \left[\sum_{i=0}^{m} (-1)^i \binom{m}{i} z^{ik} \sum_{j=0}^{\infty} \binom{m+j-1}{j} z^j \right] \text{[since } (-1)^{2j} = 1\text{]}$$

or
$$P(z) = (1-\rho k) \sum_{m=0}^{\infty} \rho^m \sum_{j=0}^{\infty} \sum_{i=0}^{m} (-1)^i \binom{m}{i}\binom{m+j-1}{j} z^{m+ik+j}$$

or
$$\sum_{n=0}^{\infty} P_n z^n = (1-\rho k) \sum_{m=0}^{\infty} \rho^m \sum_{j=0}^{\infty} \sum_{i=0}^{m} (-1)^i \binom{m}{i}\binom{m+j-1}{j} z^{m+ik+j} \qquad \text{...(23.116)}$$

Comparing the coefficients of z^n from both sides of eqn. (23.116)

$$P_n = (1-\rho k) \sum_{ijm} \rho^m (-1)^i \binom{m}{i}\binom{m+j-1}{j} \qquad \text{...(23.117)}$$

where the summation is taken over all combinations of i, j and m satisfying $n = m + ik + j$ for given k. For simplicity, we use only the *single summation* $(\sum_{ijm})$.

For example, if we take $k = 2$, $n = 7$, then

$$P_7 = (1-2\rho)(4\rho^4 + 10\rho^5 + 6\rho^6 + \rho^7).$$

(c) To obtain formula for L_s, W_s, L_q, W_q.

First, we shall determine the expected number of phases (not units in the system) denoted by $L_s^{(p)}$. Later, number of phases can be converted into units dividing by k (the number of phases in one unit). Mathematically,

$$L_s^{(p)} = \sum_{n=0}^{\infty} nP_n$$

Now, to determine $\sum_{n=0}^{\infty} nP_n$, consider the equation (23.107*c*) *i.e.*,

$$(1+\rho) P_n = P_{n+1} + \rho P_{n-k}, \; n \ge 1$$

Multiplying by n^2 and taking summation on both sides from 1 to ∞,

$$(1+\rho) \sum_{n=1}^{\infty} n^2 P_n = \sum_{n=1}^{\infty} n^2 P_{n+1} + \rho \sum_{n=1}^{\infty} n^2 P_{n-k}$$

or
$$(1+\rho) \sum_{n=1}^{\infty} n^2 P_n = \underbrace{\sum_{n=0}^{\infty} n^2 P_{n+1}}_{\text{(I)}} + \rho \underbrace{\sum_{n=k}^{\infty} n^2 P_{n-k}}_{\text{(II)}},$$

$$\left[\text{since} \sum_{n=1}^{\infty} n^2 P_{n+1} = \sum_{n=0}^{\infty} n^2 P_{n+1}, \text{ and } P_{n-k} = 0 \text{ for all } n < k\right]$$

Let $n \equiv n-1$ in summation **(I)**, $n \equiv n+k$ in summation **(II)** and altering the corresponding limits, the equation becomes

$$(1+\rho) \sum_{n=1}^{\infty} n^2 P_n = \sum_{n=1}^{\infty} (n-1)^2 P_n + \rho \sum_{n=0}^{\infty} (n+k)^2 P_n$$

or
$$(1+\rho) \sum_{n=0}^{\infty} n^2 P_n = \left[\sum_{n=0}^{\infty} (n-1)^2 P_n - P_0 \right] + \rho \sum_{n=0}^{\infty} (n+k)^2 P_n$$

$$= \sum_{n=0}^{\infty} [\{(n^2 - 2n + 1) + \rho(n^2 + k^2 + 2kn)\} P_n - P_0]$$

$$= \sum_{n=0}^{\infty} [(1+\rho) n^2 - 2n(1-k\rho) + (\rho k^2 + 1)] P_n - P_0$$

or $$(1+\rho)\sum_{n=0}^{\infty} n^2P_n = (1+\rho)\sum_{n=0}^{\infty} n^2P_n - 2(1-k\rho)\sum_{n=0}^{\infty} nP_n + (\rho k^2+1)\sum_{n=0}^{\infty} P_n - P_0$$

or $$2(1-k\rho)\sum_{n=0}^{\infty} nP_n = \rho k^2 + 1 - P_0 \qquad (\because \sum_{n=0}^{\infty} P_n = 1)$$

or $$\sum_{n=0}^{\infty} nP_n = \frac{\rho k^2 + (1-P_0)}{2(1-k\rho)}$$

Since $P_0 = 1 - k\rho$ from equation (23.114), therefore

$$L_s^{(p)} = \sum_{n=0}^{\infty} nP_n = \frac{\rho k^2 + \rho k}{2(1-\rho k)} \text{ or } = \frac{\rho k(k+1)}{2(1-\rho k)}$$

Now, expected number of units (not phases) in the queue is given by

$$L_q = \frac{L_s^{(p)} - \text{Expected number of phases in service}}{k}$$

or $$L_q = \frac{1}{k}\left[\frac{\rho k(k+1)}{2(1-\rho k)} - \frac{\lambda}{2\mu}(k+1)\right], = \frac{(k(1+k))}{2}\cdot\frac{\rho^2}{1-\rho k}$$

Since $1/\mu$ is the mean service time per unit and average number of phases of one unit in service is equal to $(k+1)/2$, time taken for service $= (k+1)/2\mu$. Hence, the expected number of phases arrived during time $(k+1)/2\mu$ will be equal to $\left(\frac{k+1}{2\mu}\cdot\lambda\right)$.

Therefore, $$L_q = \frac{k(1+k)}{2}\cdot\frac{\rho^2}{1-\rho k} = \frac{k+1}{2k}\cdot\frac{\lambda^2}{\mu(\mu-\lambda)} \qquad ...(23.118)$$

$$W_q = \frac{L_q}{\lambda} = \frac{k+1}{2k}\cdot\frac{\lambda}{\mu(\mu-\lambda)} \qquad ...(23.119)$$

$$L_s = L_q + \frac{\lambda}{\mu} = \frac{k+1}{2k}\cdot\frac{\lambda^2}{\mu(\mu-\lambda)} + \frac{\lambda}{\mu} \qquad ...(23.120)$$

$$W_s = \frac{L_s}{\lambda} = \frac{k+1}{2k}\cdot\frac{\lambda}{\mu(\mu-\lambda)} + \frac{1}{\mu}. \qquad ...(23.121)$$

23.16–2. Model V (B). $(M \mid E_k \mid 1) : (1 \mid FCFS)$

This model differs from **Model V(A)** in the sense that the capacity of the system is unity. That is, there is no queue and the system contains only one customer.

Let the customer be running in the nth phase of service such that $1 < n < k$.

Like **Model V(A)**, the following set of *steady-state difference equations* govern this model :

$$0 = -\lambda P_0 + k\mu P_1, \qquad n = 0$$
$$0 = -k\mu P_k + \lambda P_0, \qquad n = k.$$
$$0 = -k\mu P_n + k\mu P_{n+1}, \qquad 1 \le n < k.$$

From these equations, we get the following relations :

$$P_1 = \frac{\lambda}{k\mu}P_0 \text{ for } n = 0,\ P_k = \frac{\lambda}{k\mu}P_0 \text{ for } n = k \text{ and } P_{n+1} = P_n \text{ for } 1 \le n < k.$$

We observe that, $P_1 = P_2 = P_3 = \dots, = P_{k-1} = P_k$, for $n = 1, 2, \dots, k-1$.

Thus, $$P_i = \frac{\lambda}{k\mu}P_0, \text{ for } i = 1, 2, \dots, k$$

To obtain the value of P_0, we make use of the fact that $\sum_{i=0}^{k} P_i = 1$. *i.e.*,

$$P_0 + \sum_{i=1}^{k} P_i = 1 \quad \text{or} \quad P_0 + \sum_{i=1}^{k} \frac{\lambda}{k\mu} P_0 = 1 \quad \text{or} \quad P_0 = \left[1 + \frac{\lambda}{\mu} \sum_{i=1}^{k} \frac{1}{k}\right]^{-1} = \left(1 + \frac{\lambda}{\mu}\right)^{-1}$$

Hence $P_n = \frac{1}{k}\left(\frac{\lambda}{\lambda+\mu}\right)$.

Q. 1. Describe the use of Erlangian distribution in queueing process.
2. Explain Erlang's queueing model and derive an expression for the expected number in the queue.
3. Obtain the solution of M | E_x | 1 : (∞ | FIFO) queueing system. **[Garhwal M.Sc. (Maths.) 95]**
4. Obtain the steady-state solution for the number of units in the system for the queueing model M | E_x | 1 : (∞ | FCFS). **[Delhi MA/M.Sc. (OR) 92]**

23.16–3. Illustrative Examples on Model V

Example 36. *Prove that for the Erlang distribution with parameters* μ *and* k, *the mode is at* $(1 - 1/k)(1/\mu)$, *the mean is* $1/\mu$, *and the variance is* μ^2/k.

Solution. The probability density function of the *Erlangian* service time distribution is defined by

$$s(t;\mu,k) = C_k t^{k-1} e^{-k\mu t}, \; k = 1, 2, \ldots$$

where $0 \le t \le \infty$ and C_k are constants.

To determine C_k use the property : $\int_0^\infty s(t;\mu,k)\,dt = 1$ *i.e.,* $\int_0^\infty C_k t^{k-1} e^{-k\mu t}\,dt = 1$

or
$$\frac{C_k}{(k\mu)^k}\int_0^\infty z^{k-1} e^{-z}\,dz = 1 \text{ for } k\mu t = z \quad \text{or} \quad \frac{C_k}{(k\mu)^k}\Gamma k = 1 \quad \text{or} \quad C_k = \frac{(k\mu)^k}{\Gamma k} = \frac{(k\mu)^k}{(k-1)!}$$

$$\therefore \quad s(t;\mu,k) = \frac{(k\mu)^k t^{k-1} e^{-k\mu t}}{(k-1)!}, \; k = 1, 2, 3, \ldots$$

(i) **Mode :**

$$\frac{d}{dt}[s(t;\mu,k)] = 0 \text{ yields}$$

$$\frac{(k\mu)^k}{(k-1)!}\left[(k-1)t^{k-2}e^{-k\mu t} + t^{k-1}(-k\mu)e^{-k\mu t}\right] = 0 \quad \text{or} \quad t = \frac{k-1}{k\mu}.$$

Since $\frac{d^2}{dt^2}[s(t;\mu,k)]$ is negative for $t = \frac{k-1}{k\mu}$, the mode lies at

$$t = \frac{k-1}{k\mu} = \left(1 - \frac{1}{k}\right)\left(\frac{1}{\mu}\right).$$

Proved.

(ii) **Mean of *k*th Erlang :**

$$E(t) = \int_0^\infty t\,s(t;\mu,k)\,dt = \frac{(k\mu)^k}{(k-1)!}\int_0^\infty t^k e^{-\mu k t}\,dt$$

$$= \frac{(k\mu)^k}{(k-1)!}\int_0^\infty \frac{z^k}{(k\mu)^k}\frac{e^{-z}}{k\mu}\,dz = \frac{1}{k\mu(k-1)!}\Gamma(k+1) = \frac{1}{\mu} \quad \left[\text{for } k\mu t = z \text{ and } dt = \frac{dz}{k\mu}\right]$$

This proves that every member of *Erlang* family shares the common mean $1/\mu$.

(iii) **Variance of *k*th Erlang :**

$$\text{Var}(t) = [E(t^2)] - [E(t)]^2,$$

where
$$E(t^2) = \int_0^\infty t^2 s(t;\mu,k)\,dt = \frac{(k\mu)^k}{(k-1)!}\int_0^\infty t^{k+1}e^{-\mu k t}\,dt = \frac{k+1}{k\mu^2}.$$

Hence
$$\text{Var}(t) = \frac{k+1}{k\mu^2} - \frac{1}{\mu^2} = \frac{1}{k\mu^2}.$$

Proved.

Example 37. *A hospital clinic has a doctor examining every patient brought in for a general checkup. The doctor averages 4 minutes on each phase of the checkup although the distribution of time spent on each phase is approximately exponential. If each patient goes through four phases in the checkup and if the arrivals of the patients to the doctor's clinic are approximately Poisson at an average rate of 3 per hour, then what is the average time spent in the examination ? What is the most probable time spent in examination ?*

[Meerut (Maths.) 99, (Stat.) 95, 90]

Solution. Here, $k = 4$, mean service time $= 1/\mu = 4 \times 4 = 16$ minutes, and

mean inter-arrival time $= 1/\lambda = 60/3 = 20$ minutes.

The average time spent in doctor's clinic is given by the formula

$$W_q = \frac{k+1}{2k} \cdot \frac{\lambda}{\mu(\mu - \lambda)} \cdot \qquad \text{[from eqn. (23.119)]}$$

Substituting $k = 4$, $\mu = 1/16$, $\lambda = 1/20$,

$$W_q = \frac{4+1}{2 \times 4} \cdot \frac{1/20}{1/16 \times (1/16 - 1/20)} = 40 \text{ minutes.}$$

Also, mean time spent in the examination $= 1/\mu = 16$ minutes. **Ans.**

The most probable time spent in the examination is the modal value of t for the fourth member of the family, *i.e.*

$$\frac{k-1}{k\mu} = \frac{4-1}{4 \times 1/16} = 12 \text{ minutes.} \qquad \textbf{Ans.}$$

Example 38. *In a car manufacturing plant, a loading crane takes exactly 10 minutes to load a car into a wagon and again comes back to the position to load another car. If the arrival of cars is a Poisson stream at an average rate of one after every 20 minutes, calculate the average time of a car in a stationary state.*

Solution. Since the loading crane takes exactly 10 minutes to load a car, service time is constant which implies $k \to \infty$ (by property of *Erlang* Distribution)

Hence, for $k \to \infty$, the formula

$$W_q = \frac{k+1}{2k} \cdot \frac{\lambda}{\mu(\mu - \lambda)}$$

reduces to

$$W_q = \frac{1}{2}\left[1 + \frac{1}{\infty}\right]\frac{\lambda}{\mu(\mu - \lambda)} = \frac{\lambda}{2\mu(\mu - \lambda)}$$

Now, substituting the values $\lambda = 3$ per hour, $\mu = 6$ per hour,

$$W_q = \frac{3}{2 \times 6(6-3)} \times 60 \text{ minutes.} = 5 \text{ minutes.} \qquad \textbf{Ans.}$$

Example 39. *A colliery working one shift per day uses a large number of locomotives which breakdown at random intervals; on average one fails per 8 hour shift. The fitter carries out a standard maintenance schedule on each faulty loco. Each of the five main parts of this schedule take an average half an hour but the time varies widely. How much time will the fitter have for the other tasks and what is the average time a loco is out of service ?*

Solution. Since one locomotive breaks down during 8 hours, the arrival rate will be $\lambda = 1/8$ per hour.

In this problem the service time for each part is not known exactly but it can be assumed to follow an exponential distribution for each part of the schedule taking on average half an hour for each part. So the whole schedule of 5 parts will have a servicing time with *Erlang* distribution with an average of $(5 \times 1/2)$ hours as service time. Therefore, $\mu = 2/5$ per hour.

Here the arrival rate is one per 8-hour shift and the fitter takes on an average 5/2 hours for the repair of a locomotive. Hence the time taken by the fitter for the other task will be $= 8 - 5/2 = 5.5$ hours.

Therefore, the average time for the locomotive to be out of service becomes

$$W_s = \frac{k+1}{2k} \cdot \frac{\lambda}{\mu(\mu - \lambda)} + \frac{1}{\mu} = \left(\frac{5+1}{10}\right)\frac{1/8}{2/5(2/5 - 1/8)} + \frac{5}{2} = 2.18 \text{ hours.}$$

Example 40. *An airline maintenance base has facilities for overhauling only one airplane engine at a time. Therefore, in order to return the airplanes into use as soon as possible, the policy has been to stagger the* *overhauling of the four engines of each airplane. In other words, only one engine is overhauled each time an airplane comes into the shop. Under this policy airplanes have arrived according to a Poisson process at a mean rate of one per day. The time required for an engine overhaul (once work has begun) has an exponential distribution with a mean of 1/2 day.*

A proposal has been made to change the policy so as to overhaul all four engines; consequently, each time an airplane comes into the shop. It is pointed out that, although this would quadruple the expected service time, each plane would need to come into the shop only one-fourth time as often. Use qeueing theory to compare the two alternatives on a meaningful basis.

Solution. Obviously the service costs are same in both the alternatives so there must be a difference in the costs of waiting of the airplanes. Both alternatives having different mean arrival rates, the expected cost of waiting per unit time is a fixed multiple of queue length, but not of the waiting time. Therefore, the comparison of two alternatives here means the comparison of expected line lengths in the two alternatives.

First Alternative **[Model ($M \mid M \mid 1$)] :** Here $\lambda = 1$ airplane per day, $\mu = 2$ airplane per day

Average number of airplanes in the system are given by

$$L_s^{(1)} = \frac{\lambda}{\mu - \lambda} = \frac{1}{2-1} = 1.$$

Second Alternative **[Model ($M \mid E_k \mid 1$)].** Here $k = 4, \lambda = 1/4$ airplanes/day.

Since the mean service time per airplane is given by ($\frac{1}{2} \times 4$) *i.e.* 2 days, therefore $\mu = \frac{1}{2}$ airplane per day.

Hence, average number of airplanes in the system are given by

$$L_s^{(2)} = \frac{k+1}{2k} \cdot \frac{\lambda^2}{\mu(\mu - \lambda)} + \frac{\lambda}{\mu} = \frac{4+1}{8} \cdot \frac{1/16}{\frac{1}{2}(\frac{1}{2} - \frac{1}{4})} + \frac{1}{2} = 13/16 = 0.81.$$

Conclusion. Since $L_s^{(2)} = (0.81) < L_s^{(1)}$ (= 1), the waiting cost is reduced in the second alternative and hence the proposal given in the problem should be accepted.

EXAMINATION PROBLEMS (ON MODEL V)

1. A barber with one man shop takes exactly 25 minutes to complete one hair cut. If customers arrive in a Poisson fashion at an average rate of one every 40 minutes, how long on the average must a customer wait for service. Also, find the average time a customer spends in the barber shop. **[Meerut (M.Sc. Maths) 2003, 90]**

   [**Hint.** Here $\lambda = 1/40$, $\mu = 1/25$, $k \to \infty$. Use formula : $W_q = \lim_{k \to \infty} \frac{k+1}{2k} \cdot \frac{\lambda}{\mu(\mu - \lambda)}$.]

   [**Ans.** $W_q = 20.8$ minutes, $W_s = W_q + 1/\mu = 45.8$ minutes.]

2. At a certain airport it takes exactly 5 minutes to land an airplane once it is given the signal to land. Although incoming planes have scheduled arrival times the wide variability in arrival times, it produces an effect which makes the incoming planes appear to arrive in a Poisson fashion at an average rate of six per hour. This produces occasional stack-ups at the airport which can be dangerous and costly. Under these circumstances, how much time will a pilot expect to spend circling the field to land ?

   [**Hint.** Here $\lambda = 1/10$, $\mu = 1/5$, $k \to \infty$. Use the formula : $W_q = \lim_{k \to \infty} \frac{k+1}{2k} \cdot \frac{\lambda}{\mu(\mu - \lambda)}$.]

   [**Ans.** $W_q = 2.5$ minutes]

3. Repairing a certain type of machine which breaks down in a given factory consists of *five basic steps* that must be performed sequentially. The time taken to perform each of the given steps is found to have an exponential distribution with mean 5 minutes and is independent of the other steps. If these machines break down in a Poisson fashion at an average rate of two per hour, and if there is only one repairman, what is the average idle time for each machine that has broken down ? **[Garhwal M.Sc. (Math.) 95]**

   [**Hint.** Here $\lambda = 2/60$, $\mu = \frac{1}{5 \times 5}$, $k = 5$, Use the formula $W_s = \frac{k+1}{2k} \cdot \frac{\lambda}{\mu(\mu - \lambda)} + \frac{1}{\mu} = (75 + 25)$ minutes.]

   [**Ans.** $W_s = 100$ minutes].

4. A warehouse in a small state receives orders for a certain item and sends them by a truck as soon as possible to the customer. The orders arrive in the Poisson fashion at a mean rate of 0.9 per day. Only one item at a time can be shipped by truck from the warehouse which is located in the central part of the state, the distribution of service time in days has a distribution with probability density $4te^{-2t}$, what is the expected delay between the arrival of an order and the arrival of the item to the customer ? Service time here implies the time the truck takes to load, get to the customer, unload and return to the warehouse. Loading and unloading times are small compared with travel time.

   [**Hint.** In this problem, $s(t; \mu, k) = \frac{(k\mu)^k}{(k-1)!} t^{k-1} e^{-k\mu t} = 4te^{-2t}$. This gives $\mu = 1$ and $k = 2$. This indicates that the service time distribution is a second member of Erlang family with $\mu = 1$. Also, $\lambda = 0.9$ customer per day. Now use formula for W_q and W_s.]

   [**Ans.** $W_q = 6.75$ days, $W_s = W_q + 1/\mu = 7.75$ days.]

5. A barber runs his own saloon. It takes exactly 25 minutes to complete one haircut. Customers arrive in a Poisson fashion at an average rate of one every 35 minutes.
(a) For what per cent of time would the barber be idle ? (b) What is the average time of a customer spent in the spop ?
[Hint. Here $\lambda = 60/35$ per hour, $\mu = 60/25$ per hour.
(a) Find $P_0 = 1 - \lambda/\mu = 1 - 0.71 = 0.29$, (b) Find $W_q = \lim_{k \to \infty} \frac{k+1}{2k} \cdot \frac{\lambda}{\mu(\mu - \lambda)}$.]
[**Ans.** (a) 29%, (b) 30 minutes]

23.17. MODEL VI. MACHINE SERVICING MODEL $(M \mid M \mid R) : (K \mid K \mid GD)$, $K > R$

In this queueing system, there are K machines which are serviced by R repairmen. Whenever a machine breaks down, it will result in a loss of production unit it is repaired. Consequently, a broken machine cannot generate new calls while in service. This is equivalent to the finite calling source with maximum limit of K potential customers.

In this model, the approximate probability of a single service during an instant Δt is $n\mu\Delta t$ for $n \le R$, and $R\mu\,\Delta t$ for $n \ge R$. On the other hand, the probability of a single arrival during Δt is approximately $(K-n)\,\lambda\,\Delta t$ for $n \le K$, where λ is defined in this case as the rate of breakdown per machine. Thus,

$$P_0(t+\Delta t) = P_0(t)(1 - K\lambda\Delta t) + P_1(t)\,\mu\Delta t\,\{1-(K-1)\lambda\Delta t\}, \text{ for } n = 0,$$

$$P_n(t+\Delta t) = P_n(t)\{1-(K-n)\lambda\Delta t\}(1-n\mu\Delta t) + P_{n-1}(t)\{(K-n+1)\lambda\Delta t\}\{1-(n-1)\mu\Delta t\}$$
$$+ P_{n+1}(t)\{1-(K-n-1)\lambda\Delta t\}\{(n+1)\mu\Delta t\}, \text{ for } 0 < n < R$$

$$P_R(t+\Delta t) = P_R(t)\{1-(K-R)\lambda\Delta t\}(1-R\mu\Delta t) + P_{R+1}(t)[1-\{K-(R+1)\}\lambda\Delta t](R\mu\Delta t)$$
$$+ P_{R-1}(t)\{K-(R-1)\lambda\Delta t\}\{1-(R-1)\mu\Delta t\}, \text{ for } n = R$$

$$P_n(t+\Delta t) = P_n(t)\{1-(K-n)\lambda\Delta t\}(1-R\mu\Delta t) + P_{n-1}(t)\{(K-n+1)\lambda\Delta t\}(1-R\mu\Delta t)$$
$$+ P_{n+1}(t)\{1-(K-n-1)\lambda\Delta t\}(R\mu\Delta t), \text{ for } R < n \le K-1$$

$$P_K(t+\Delta t) = P_K(t)(1-R\mu\Delta t).1 + P_{K-1}(t)\{\lambda\Delta t(1-R\mu\Delta t)\}, \text{ for } n = K$$

(i) To obtain steady state equations :

Adopting the same method as discussed in the earlier models, the steady state equations are given by

$$K\rho P_0 = P_1, \text{ for } n = 0$$
$$\{(K-n)\rho + n\}P_n = (K-n+1)\rho P_{n-1} + (n+1)P_{n+1}, \text{ for } 0 < n < R,$$
$$\{(K-n)\rho + R\}P_n = (K-n+1)\rho P_{n-1} + RP_{n+1}, \text{ for } R < n \le K-1,$$
$$RP_K = \rho P_{K-1}, \text{ for } n = K.$$

(ii) To solve the system of steady state equations :

From the first steady state difference equation, $P_1 = K\rho P_0$.

Substituting $n = 1$ in the second difference equation, $2P_2 = (K-1)\rho P_1$.

By induction method, it can be shown that, $(n+1)P_{n+1} = (K-n)\rho P_n$, $0 \le n \le R$.

Similarly, from the remaining two difference equations, $RP_{n+1} = (K-n)\rho P_n$, for $R \le n \le K$.

The last two equations can be verified to give the required solution

$$P_n = \begin{cases} \binom{K}{n}\rho^n P_0, & 0 \le n \le R, \\ \binom{K}{n}\dfrac{n!\,\rho^n}{R!\,R^{n-R}} P_0, & R \le n \le K. \end{cases}$$

where $$P_0 = \left\{\sum_{n=0}^{R}\binom{K}{n}\rho^n + \sum_{n=R+1}^{K}\binom{K}{n}\frac{n!\,\rho^n}{R!\,R^{n-R}}\right\}^{-1}$$

The other measures are obtained as follows :

$$L_q = \sum_{n=R+1}^{K}(n-R)P_n = \sum_{n=0}^{K} nP_n - \left\{R - \sum_{n=0}^{R}(R-n)P_n\right\} = L_s - (R - \bar{R})$$

or $$L_s = L_q + (\bar{R}) \text{ where } \bar{R} = \text{expected number of idle repairmen} = \sum_{n=0}^{R}(R-n)P_n.$$

In this queueing system, the arrivals occur with a rate λ but all arrivals do not join the system (for example, in the cases where the maximum allowable queue length is reached no new arrivals are allowed to join the queue). However, by defining λ to include only those arrivals that join the system, these relationships can be made to hold. Suppose λ_{eff} define the effective arrival rate, value of λ_{eff} can be conveniently determined from

$$L_q = L_s - \frac{\lambda_{eff}}{\mu} \quad \text{or} \quad \lambda_{eff} = \mu\,[L_s - L_q].$$

Once λ_{eff} is known, then from general formulae

$$W_q = L_q/\lambda_{eff},\ W_s = L_s/\lambda_{eff}.$$

Q. Using theory of steady state equation, write a short note on machine repairing problem.

[Garhwal M.Sc. (Stat.) 96, 95, 93, 91]

Example 41. (a) *There are five machines, each of which, when running, suffers break downs at an average rate of 2 per hour. There are two servicemen and only one man can work on a machine at a time. If n machines are out of order when n > 2, then (n – 2) of them wait until a serviceman is free. Once a serviceman starts work on a machine, the time to complete the repair has an exponential distribution with mean 5 minutes. Find the distribution of the number of machines out of action at a given time.*

(b) *Find the average time an out-of-action machine has to spend waiting for the repairs to start.*

Solution. We are given that

K = total number of machines in the sytem = 5, R = number of servicemen = 2, $\lambda = 2$ and $\mu = 60/5 = 12$.

Let n = number of machines out of order. Using formula for P_n, we have

$$P_n = \begin{cases} \binom{5}{n}\left(\frac{2}{12}\right)^n P_0, & 0 \le n < 2 \\ \binom{5}{n}\frac{n!}{2^{n-2}\,2!}\left(\frac{2}{12}\right)^n P_0, & 2 \le n \le 5 \end{cases}$$

and

$$P_0 = \left[\sum_{n=0}^{2-1}\binom{5}{n}\left(\frac{2}{12}\right)^2 + \sum_{n=2}^{5}\binom{5}{n}\frac{n!}{2^{n-2}\,2!}\left(\frac{2}{12}\right)^n\right]^{-1}$$

$$= 648/1493 \qquad \text{(after simplification)}$$

Substituting this value of P_0 in P_n, we get

$$P_n = \begin{cases} \binom{5}{n}\left(\frac{1}{6}\right)^n \frac{648}{1493}, & 0 \le n < 2 \\ \binom{5}{n} 2\,(n\,!)\left(\frac{1}{12}\right)^n \frac{648}{1493}, & 2 \le n \le 5 \end{cases}$$

Average number of machines out of action is given by

$$L_q = \sum_{n=2+1}^{5}(n-2)\,P_n = \sum_{n=3}^{5}(n-2)\,P_n = P_3 + 2P_4 + 3P_5 = \frac{165}{1493}.$$

Average time an out-of-action machine has to spend waiting for the repairs to start is, $W_q = L_q/\lambda_{eff}$

But, $\quad \lambda_{eff} = \sum_{n=0}^{5}(5-n)\,P_n = \lambda\,[5P_0 + 4P_1 + 3P_2 + 2P_3 + P_4] = 3\lambda \times \frac{2050}{1493} = 6 \times \frac{2050}{1493} \qquad (\because \lambda = 2)$

$\therefore \quad W_q = \frac{165}{1493} \times \frac{1493}{6 \times 2050} = \frac{55}{4100}$ hrs. $= \frac{33}{41}$ minutes. **Ans.**

EXAMINATION PROBLEMS

1. A repairman is to be hired to repair machines which breakdown at an average rate of 6 per hour. The breakdowns follow Poisson distribution. The productive time of a machine is considered to cost Rs. 20 per hour. Two repairmen, Mr. X and Mr. Y have been interviewed for this purpose. Mr. X charges Rs. 10 per hour and he services breakdown machines at the rate of 8 per hour. Mr. Y demands Rs. 14 per hour and he services at an average rate of 12 per hour. Which repairman should be hired ? (Assume 8 hours shift per day)

[Ans. Mr. Y should be hired since the total cost involved is Rs. 272, whereas the cost involved in the case of repairman X is Rs. 560]

2. A mechanic services four machines. For each machine. the mean time between sevice requirements is 10 hours and is assumed to be from an exponential distributions. The repair time tends to follow the same distribution with a mean of two hours. When a machine is down for repairs, the time lost has a value of Rs. 20 per hour. The mechanic costs Rs. 50 per day. Find :
   (i) What is the expected number of machines in operation ? (ii) What is the expected down time cost per day ?
   (iii) Would it be desirable to provide two mechanics each to service only two machines.
   [Ans. (i) 3 machines, (ii) Rs. 200/– per day, (iii) Not economical to engage two mechanics.]

23.18. MODEL VII (POWER SUPPLY MODEL)

In this model suppose there is an electric circuit supplying power to 'c' consumers. The requirements of consumers follow Poisson distribution with parameter λ, and the supply schedule also follows Poisson distribution with parameter μ.

If there are n customers in the queue at any time t, then

$$\left.\begin{aligned}\lambda_n &= (c-n)\,\lambda \\ \mu_n &= n\mu\end{aligned}\right\} 0 \le n \le c$$

and the equations governing this system are obtained as follows :

$$P_0\,(t+\Delta t) = P_0\,(t)\,(1-\lambda_0\Delta t) + P_1(t)\,\mu_1\,\Delta t + O\,(\Delta t)^2\,,\ \text{for } n=0$$
$$= P_0\,(t)\,(1-c\lambda\Delta t) + P_1(t)\,\mu\Delta t + O\,(\Delta t)^2$$
$$P_n\,(t+\Delta t) = P_n\,(t)\,(1-\lambda_n\Delta t)\,(1-\mu_n\Delta t) + P_{n-1}\,(t)\,\lambda_{n-1}\,\Delta t + P_{n+1}\,(t)\,\mu_{n+1}\,\Delta t$$
$$= P_n\,(t)\,[1-(\mu_n+\lambda_n)\,\Delta t] + P_{n-1}\,(t)\,\lambda_{n-1}\,\Delta t + P_{n+1}\,(t)\,\mu_{n+1}\,\Delta t$$
$$= P_n\,(t)\,[1-\{(c-n)\,\lambda + n\mu\}\,\Delta t] + P_{n-1}\,(t)(c-n+1)\,\lambda\Delta t + P_{n+1}(t)(n+1)\,\mu\Delta t\,,$$

for~1~<=~ n ~<=~c ~--~1

and $P_c\,(t+\Delta t) = P_c\,(t)\,(1-c\mu\Delta t) + P_{c-1}\,(t)\,\lambda\Delta t$ for $n=c$.

Thus, taking appropriate limits for steady state system, to obtain difference equations

$$c\lambda P_0 = \mu P_1\,,\ \text{for } n=0$$
$$[(c-n)\,\lambda + n\mu]\,P_n = (c-n+1)\,\lambda P_{n-1} + (n+1)\,\mu P_{n+1}\,,\ \text{for } 1 \le n \le c-1$$

and

$$c\mu P_c = \lambda P_{c-1}\,,\ \text{for } n=c.$$

From these equations, the recurrence relation is given by

$$(c-n)\,\lambda P_n = (n+1)\,\mu P_{n+1}\,,\ n=0,1,\ldots,c.$$

Therefore,

$$P_1 = c\,(\lambda/\mu)\,P_0\,,\ \text{for } n=0$$
$$P_2 = \frac{c\,(c-1)}{2\,!}\left(\frac{\lambda}{\mu}\right)^2 P_0\,,\quad \text{for } n=1$$
$$\vdots$$
$$P_n = \frac{c\,(c-1)\,(c-2)\ldots(c-n+1)}{n\,!}\left(\frac{\lambda}{\mu}\right)^n P_0\,,$$

and

$$P_c = \left(\frac{\lambda}{\mu}\right)^c P_0$$

Since $\sum_{n=0}^{c} P_n = 1$, *i.e.*,

$$P_0\left[1 + \frac{c\lambda}{\mu} + \frac{c\,(c-1)}{2\,!}\left(\frac{\lambda}{\mu}\right)^2 + \ldots + \left(\frac{\lambda}{\mu}\right)^c\right] = 1 \ \text{ or } \ P_0\left(1+\frac{\lambda}{\mu}\right)^c = 1 \ \text{ or } \ P_0 = \left(\frac{\mu}{\lambda+\mu}\right)^c$$

Thus,

$$P_n = \frac{c\,(c-1)\ldots(c-n+1)}{n\,!}\left(\frac{\lambda}{\mu}\right)^n\left(\frac{\mu}{\lambda+\mu}\right)^c = \binom{c}{n}\left(\frac{\lambda}{\lambda+\mu}\right)^n\left(\frac{\mu}{\lambda+\mu}\right)^{c-n}$$

which is a binomial distribution.

23.19. MODEL VIII (ECONOMIC COST PROFIT MODEL)

Consider the single station model with an arrival rate λ and service rate μ. Assume that service rate μ can be controlled and it is required to determine its optimum value based on an approximate cost model.

Let c_1 = cost per service per unit time, c_2 = cost of waiting per unit time for each customer,

$C(\mu)$ = total cost of waiting plus service per unit time, given the service rate μ.

Then,
$$C(\mu) = c_1\mu + c_2 L_s. \quad \text{...(23.122)}$$

Since the service rate μ is continuous, its optimum value can be obtained by differentiating the cost function $C(\mu)$ with respect to 'μ'.

As a special case, for $(M \mid M \mid 1) : (\infty \mid FCFS)$

$$C(\mu) = c_1\mu + c_2 L_s \text{ or } C(\mu) = c_1\mu + c_2\left(\frac{\lambda}{\mu - \lambda}\right) \text{[from eqn. (23.63)]} \quad \text{...(23.123)}$$

From necessary condition for maxima or minima

$$\frac{dC}{d\mu} = c_1 - \frac{\mu c_2}{(\mu - \lambda)^2} = 0 \text{ which yields the optimum value}$$

$$\mu^* = \lambda + \sqrt{(\lambda c_2/c_1)} \quad \text{...(23.124)}$$

provided $d^2C/d\mu^2$ is positive for minimum cost. Obviously, $d^2C/d\mu^2 = 2\lambda c_2/(\mu - \lambda)^3$, which is positive if $\mu > \lambda$.

The equation (23.124) shows that the optimum value of the parameter μ is not only dependent on c_1 and c_2, but on arrival rate λ also. This also seems logical, because μ is independent of λ. It may cause the traffic intensity $\rho = \lambda/\mu$ to be greater than unity since the condition for minimum cost is $\mu > \lambda$.

To determine the Optimum Number of Servers.

Consider the multiple server model $(M \mid M \mid s) : (\infty \mid FCFS)$.

In this case, a cost model for determining the optimum number of servers s can be formed. Assume λ and μ are fixed thereby the cost equation becomes

$$C(s) = c_1 s + c_2 L_s(s) \quad \text{...(23.125)}$$

where c_1 = cost per server per unit time, c_2 = cost of waiting per unit time for each server.

$L_s(s)$ = expected number of customers in the system when there are s number of servers.

Since the number of servers 's' can be measured in *discrete* units only, the *difference method* can be applied (*differentiation* is not applicable here). The condition for minimum of the cost function $C(s)$ is given by

$$\Delta C(s-1) < 0 < \Delta C(s)$$

Consider $\quad \Delta C(s-1) < 0 \text{ or } C(s) - C(s-1) < 0$

or $\quad sc_1 + c_2 L_s(s) < (s-1)c_1 + c_2 L_s(s-1)$

or $\quad L_s(s-1) - L_s(s) > c_1/c_2 \quad \text{...(23.126a)}$

Similarly, $0 < \Delta C(s)$ or $0 < \{C(s+1) - C(s)\}$ gives

$$sc_1 + c_2 L_s(s) < (s+1)c_1 + c_2 L_s(s+1)$$

or $\quad L_s(s) - L_s(s+1) < c_1/c_2 \quad \text{...(23.126b)}$

Combining the results (5.126a) and (5.126b), the resultant condition for minimum of $C(s)$ is

$$L_s(s) - L_s(s+1) < c_1/c_2 < L_s(s-1) - L_s(s). \quad \text{...(23.127)}$$

Example 42. *Find the optimum number of servers, given that*

$$\lambda = 1.75, \mu = 1, c_1 = 0.4, c_2 = 2.$$

Solution. Substituitng the given values in the result

$$L_s(s) = \frac{(\lambda/\mu)\, P_0}{s\,!} \cdot \frac{\rho}{(1-\rho)^2}$$

where P_0 is given by eqn. (23.96). Following table may be computed :

s	$L_s(s-1)$ (1)	$L_s(s)$ (2)	(1)–(2)		$c_1/c_2 = 0.2$	
1	—	∞	—			
2	∞	5.70	—			
3	5.23	0.47	4.76	>	0.2	
4	0.38	0.09	0.29	>	0.2	
5	0.07	0.02	0.05	<	0.2	(Note.)

From this table it is concluded that the optimum number of servers is $s = 4$.

23.20. MODEL IX $(M \mid G \mid 1) : (\infty \mid GD)$

In this model,

$M \to$ Poisson arrivals, $G \to$ general output distribution, $\infty \to$ waiting room capacity is infinite

$GD \to$ general service discipline such as *FCFS, LCFS, SIRO, etc.*

In order to determine the mean queue length and mean waiting time for this system, a technique different from that followed earlier will be used.

Let $f(t) \to$ *p.d.f* of service time distribution with mean $E(t)$ and Var $\{t\}$.

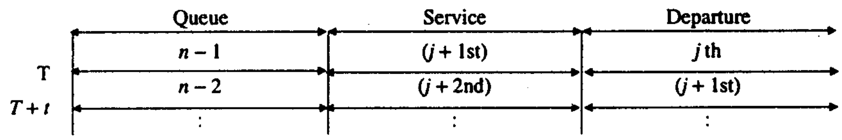
$n \to$ number of customers in the system just after a customer departs

$t \to$ time to serve the customer following the one already departed

$k \to$ number of new arrivals during t

$n' \to$ number of customer's left behind the next departing customer.

These symbols can be better explained by the following diagram, where T represents the time when the jth customer departs, and $(T+t)$ represents the time when the next $(j+1)$th customer departs. The notations $j, j+1, \ldots$ do not necessarily mean that customers are introduced into service on *FCFS* discipline. Rather, it identifies the different customers departing from the system. Thus, the result of this model can be applied to any one of the three service disciplines *FCFS, FCLS,* and *SIRO.*



The system is observed only just after a service departure has been completed. Such instant of time defines the *regeneration point.* If n customers are in the system at time T initially, then at the next moment $(T+t)$ the number n' in the system is given by

$$n' = \begin{cases} k & , \text{if } n = 0 \\ (n-1)+k, & \text{if } n > 0 \end{cases} \qquad \ldots(23.128)$$

where $k = 0, 1, 2, \ldots$, is the number of arrivals during the service time.

Alternately, if

$$\delta = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{if } n > 0 \end{cases} \qquad \ldots(23.129)$$

then
$$n' = n - 1 + \delta + k. \qquad \ldots(23.130)$$

Now taking expectation on both sides,

$$E\{n'\} = E\{n\} + E(\delta) + E(k) - 1$$

or
$$E(\delta) = 1 - E\{k\} \quad (\text{since } E(n') = E(n) \text{ in steady state}) \qquad \ldots(23.131)$$

Also, squaring both the sides on eqn. (23.130)

$$(n')^2 = [n + (k-1) + \delta]^2$$

or
$$(n')^2 = n^2 + (k-1)^2 + 2n(k-1) + \delta^2 + 2n\delta + 2k\delta - 2\delta$$

Since δ can take values 0 and 1 only, so $\delta^2 = \delta$ and $n\delta = 0$. Thus, above expression becomes

$$(n')^2 = n^2 + k^2 + 2n(k-1) + \delta(2k-1) - 2k + 1$$

or
$$2n(1-k) = n^2 - n'^2 + k^2 + \delta(2k-1) - 2k + 1$$

Now taking expectation on both sides,

$$2E(n)[1-E(k)] = E(n^2) - E(n'^2) + E(k^2) + E(\delta)E(2k-1) - 2E(k) + 1$$

or
$$E(n) = \frac{E(k^2) - 2E(k) + E(\delta)[2E(k)-1] + 1}{2[1-E(k)]}$$

$$[\because E(n^2) = E(n'^2) \text{ and } E\{\delta(2k-1)\} = E(\delta)E(2k-1)]$$

Substituting for $E(\delta) = 1 - E(k)$ from eqn. (23.131),

$$E(n) = \frac{E(k^2) - 2E(k) + [1-E(k)][2E(k)-1] + 1}{2[1-E(k)]}$$

or
$$E(n) = \frac{E(k^2) + E(k) - 2E^2(k)}{2[1-E(k)]}$$

Now, in order to determine $E(n)$ the values of $E(k)$ and $E(k^2)$ are to be computed.

Since the arrivals in time t follow the Poisson distribution,

$$E(k/t) = \lambda t, \quad E(k^2/t) = (\lambda t)^2 + \lambda t, \quad V(k/t) = E(k^2/t) - E^2(k/t).$$

Hence
$$E(k) = \int_0^\infty E(k/t) f(t)\, dt = \int_0^\infty \lambda t f(t)\, dt = \lambda E(t)$$

Also, $E(k^2) = \int_0^\infty E(k^2/t) f(t)\, dt = \int_0^\infty [(\lambda t)^2 + \lambda t] f(t)\, dt = \lambda^2 E(t^2) + \lambda E(t) = \lambda^2 V(t) + \lambda^2 E^2(t) + \lambda E(t)$

$$\{\because V(t) = E(t^2) - E^2(t)\}$$

$$\therefore \quad L_s = E(n) = \lambda E(t) + \frac{\lambda^2 [E^2(t) + V(t)]}{2[1-\lambda E(t)]} \qquad \text{...(23.132)}$$

This is called the *Pollaczek Khintchine* (*P - K*) formula.

It is noticed that $\lambda E[t] < 1$, otherwise L_s becomes negative which cannot be.

In particular, let $G = M$, then

$$\text{Var}(t) = 1/\mu^2, \; E(t) = 1/\mu,$$

$$L_s = \frac{\lambda}{\mu} + \frac{\lambda^2[(1/\mu^2) + (1/\mu^2)]}{2(1-\lambda/\mu)} = \rho + \frac{2\rho^2}{2(1-\rho)} = \frac{\rho}{1-\rho}$$

which is the same as obtained earlier for model $M \mid M \mid 1$.

Efficiency of a queueing system :

In order to measure the efficiency of a queueing system, a ratio may be defined as follows :

$$\frac{W_q}{E(t)} = \frac{\textbf{average waiting time in queue}}{\textbf{average service time}} = \frac{\textbf{useless time}}{\textbf{useful time}}$$

Thus, smaller the ratio, better will be the system.

***Alternative form* of Pollaczek formula :** Consider

$E(n)$ = average number of customers in the sytem

$E(w+t)$ = waiting time in (queue + service)

So $\quad E(n) = \lambda E(w+t) = \lambda E(w) + \lambda E(t).$ ...(23.133)

Now, equating the results (23.132) and (23.133)

$$\lambda E(w) + \lambda E(t) = \lambda E(t) + \frac{\lambda^2[E^2(t) + \text{Var}(t)]}{2[1-\lambda E(t)]}$$

or
$$\frac{E(w)}{E(t)} = \frac{\lambda[E^2(t) + \text{Var}(t)]}{2E(t)[1-\lambda E(t)]}$$

In particular, if $E(t) = 1/\mu$, then $V(t) = \sigma_t^2$ and $C_t = \sigma_t/(1/\mu)$. Therefore, this ratio becomes

$$\frac{E(w)}{E(t)} = \frac{\rho}{2(1-\rho)}(1 + C_t^2).$$

This formula was also suggested by *Pollaczek*.

For $(M \mid M \mid 1) : (\infty \mid FCFS)$ model, this ratio will be $\rho/(1-\rho)$.

The various formulae for $(M \mid G \mid 1) : (\infty \mid GD)$ can be summarized as follows :

(i) Average number of customers in the system $= \dfrac{\lambda^2 \sigma^2 + \rho^2}{2(1-\rho)} + \rho$

(ii) Average queue length

$$= \left[\frac{\lambda^2 \sigma^2 + \rho^2}{2(1-\rho)} + \rho \right] - \rho = \frac{\lambda^2 \sigma^2 + \rho^2}{2(1-\rho)}$$

(iii) Average waiting time of a customer in the queue $= \dfrac{\lambda^2 \sigma^2 + \rho^2}{2\lambda(1-\rho)}$.

(iv) Average waiting time that a customer spends in the system $= \dfrac{\lambda^2 \sigma^2 + \rho^2}{2\lambda(1-\rho)} + \dfrac{1}{\mu}$.

Q. 1. Investigate a first come first severed, single channel queue with Poisson arrivals and general service time, or describe the $M \mid G \mid 1$ queueing process analysing the steady state solutions.

2. Write short note on Pollaczek-Khintchine formula for queueing model $M \mid G \mid 1 : (\infty \mid FCFS)$.

3. Discuss the salient features of $M \mid G \mid 1$ quequeing system. **[Garhwal M.Sc. (Stat.) 92]**

4. Explain the concept of imbedded Markov chain. Find the probaility generating function (p.g.f.) for the number of units in the system (under steady state) for the queueing model $M \mid G \mid L : (\infty \mid FCFS)$. **[Delhi MA/M.Sc. (OR) 93]**

5. Obtain expected number of units in the queueing system $M \mid G \mid 1$ under steady state. **[Delhi MA/M.Sc (Stat.) 95]**

23.21. MIXED QUEUEING MODEL

If either the arrival rate or the service rate is fixed with the other one is random, then such a model is known as a *Mixed Queueing Model*

23.21–1. Model X (A) : $(M \mid D \mid 1)$

This queueing system involves a single server, Poisson arrivals, and regular or deterministic service time distribution. Here service time is not a random variable but a constant. Without any loss of generality, service time can be taken unity, *i.e.*, $\mu = 1$. Then, we can easily find P_0, P_1, P_n, and the average number of customers in the system.

(a) To find P_0 :

Let P_0 = probability that there is no unit in the system at the end of unit time.

This is possible in any of the following ways :

(i) There is no arrival during the unit time interval and there was no unit at the beginning of the interval also; *i.e.*, $P_0 = P_0 \times$ *(probability of no arrival)*.

(ii) There was one unit at the beginning, and it was served during the unit time interval and no arrival during this time period; *i.e.*,

$$P_0 = P_1 \times \text{(probability of no arrival)} \times \text{(probability that one unit is served in unit time)}$$

Since both the above possibilities are mutually exclusive, by addition law we have

$$P_0 = P_0 \times \textit{(probability of no arrival)} + P_1 \textit{ (probability of no arrival)} \times 1$$

or $$P_0 = (P_0 + P_1) \times \textit{probability of no arrival}$$

But, arrival rate is a random variable having Poisson distribution with parameter λ. Therefore,

$$P_0 = \frac{(\lambda t)^0 e^{-\lambda}}{0!} = e^{-\lambda}$$

Thus, $$P_0 = (P_0 + P_1) e^{-\lambda} = \sum_{i=0}^{1} P_i e^{-\lambda}.$$

(b) To find P_1 :

Let P_1 = probability that there is one unit in the system.

This is possible in any of the following two ways :

(i) There is no unit in the system and one unit arrives in the system, *i.e.*,

$$P_1 = P_0 \times (\textit{probability of one arrival}) = P_0 \times \lambda e^{-\lambda}.$$

(ii) There are two units at the beginning of the period and one unit is served during the unit time, and there is no arrival during this time period, *i.e.*,

$$P_1 = P_2 \times (\textit{probability of no arrival}) \times (\textit{Prob. of one service in unit time}).$$

or $$P_1 = P_2 \times e^{-\lambda} \times 1.$$

Again, both the cases are mutually exclusive. Thus, $P_1 = P_0 \lambda e^{-\lambda} \times P_2 e^{-\lambda}$.

(c) **To find P_n :**

$P_n =$ Prob. [that there are n units in the system at the end of unit time interval]

$=$ Prob. (that there were $(n+1)$ units in the system at the beginning of the period, one unit is served and there is no arrival) + Prob. (that there were $(n-1)$ units in the system at the beginning, none is served, and one arrival during the unit interval)

$= P_{n+1} \times 1 \times e^{-\lambda} + P_{n-1} \times \lambda e^{-\lambda}$.

or $$P_n = P_{n+1} e^{-\lambda} + P_{n-1} \lambda e^{-\lambda}.$$

(d) **To find L_s .**

The average number of customers in the system is given by

$$L_s = \sum_{n=0}^{\infty} nP_n = \sum_{n=0}^{\infty} n\,[P_{n+1} e^{-\lambda} + P_{n-1} \lambda e^{-\lambda}] = e^{-\lambda} \sum_{n=0}^{\infty} n\,(P_{n+1} + \lambda P_{n-1})$$

23.21-2. Model X(B). (*M* | *D* | 1) : (∞ | *FCFS*)

In this model, $M \to$ Poisson arrivals, $D \to$ deterministic service time, $1 \to$ single server, $\infty \to$ capacity of the sytem is infinite

Further assume that the steady state condition prevails with $\rho = \lambda < 1$. Let

$n \to$number of customers in the system just after a customer departs

$l \to$time of service of the customers following the one that already departed.

$k \to$number of arrivals during the service time (which is taken to be the unit of time).

$n' \to$number of customers left behind the next departing customer.

In this scheme, the system is observed only just after a service departure has been completed. Such instant of time defines the regeneration point. If n customers are in the system at observation time, then at the time of next observation the number n' in the system is given by

$$n' = \begin{cases} k \ , & \text{if } n = 0 \\ n - 1 + k, & \text{if } n > 0 \end{cases}$$

where $k = 0, 1, 2, \ldots$ is the number of arrivals during the service time.

Let q_k be the probability that k persons arrive during a service time. Since arrivals occur according to Poisson distribution with mean arrival rate λ Then

$$q_k = \frac{e^{-\lambda} (\lambda)^k}{k\,!}, \ k = 0, 1, .., \infty.$$

The transaction matrix between the end and the beginning of the service is given by

$n \backslash n' \to$	0	1	2	3	4	...	...
0	q_0	q_1	q_2	q_3	q_4	...	...
1	q_0	q_1	q_2	q_3	q_4	...	...
2	0	q_0	q_1	q_2	q_3	...	...
3	0	0	q_0	q_1	q_2	...	...
4	0	0	0	q_0	q_1	...	...
:	:	:	:	:	:		
:	:	:	:	:	:		

where q_0 = probability of no arrival during the service time.

Let p_n $(n = 0, 1, 2, \ldots)$ be the steady state probability that there are n units in the system. These probabilities are given by

$$p_0q_0 + p_1q_0 = p_0$$
$$p_0q_1 + p_1q_1 + p_2q_0 = p_1$$
$$\ldots \quad \ldots \quad \ldots$$
$$p_0q_n + p_1q_n + p_2q_{n-1} + p_3q_{n-2} + \ldots + p_{n+1}q_0 = p_n, \text{ for } n = 0, 1, 2, \ldots, \infty.$$

The last equation can now be put in the general form

$$p_n = p_0q_n + \sum_{j=1}^{n+1} p_jq_{n+1-j}, \; n = 0, 1, \ldots, \infty.$$

Applying the z-transformation* to this equation.

$$\sum_{n=0}^{\infty} z^n p_n = \sum_{n=0}^{\infty} z^n p_0q_n + \sum_{n=0}^{\infty} z^n \{\sum_{j=1}^{n+1} p_j q_{n+1-j}\}$$

$$= \sum_{n=0}^{\infty} z^n p_0q_n + \sum_{n=0}^{\infty} z^{n+1-1} \{\sum_{j=1}^{n+1} p_jq_{n+1-j}\}$$

$$= p_0 \sum_{n=0}^{\infty} z^n q_n + \frac{1}{z} \sum_{n=0}^{\infty} z^m \{\sum_{j=0}^{m} p_j q_{m-j} - p_0q_m\}, \qquad \text{where } m \equiv n+1 \text{ (odd term } j = 0)$$

$$= p_0q_0 + z(q_0p_1 + q_1p_0) + z^2(q_0p_2 + q_1p_1) + \ldots$$
$$= q_0(p_0 + p_1z + p_2z^2 + \ldots) + q_1z(p_0 + p_1z + p_2z^2 + \ldots) + q_2z^2(p_0 + p_1z + p_2z^2 + \ldots) + \ldots$$
$$= (q_0 + q_1z + q_2z^2 + \ldots)(p_0 + p_1z + p_2z^2 + \ldots)$$
$$= Z(q_n) \,.\, Z(p_n).$$

To simplify the expression, consider $r_m = \sum_{J=0}^{m} p_j q_{m-j}$

Clearly r_m is the *convolution*** of p_m and q_m. Consequently,

$$Z(r_m) = Z(q_m)\, Z(p_m)$$
$$\downarrow \quad \downarrow \quad \downarrow$$
$$R(z) \quad Q(z) \quad P(z)$$

Thus, from the transformed equation

$$P(z) = p_0\, Q(z) + \frac{1}{z}\left\{\sum_{m=1}^{\infty} z^n r_m\right\} - \frac{p_0}{z}\{Q(z) - q_0\}$$

$$= p_0\, Q(z) + \frac{1}{z}\{R(z) - r_0\} - \frac{p_0}{z}\{Q(z) - q_0\}$$

or

$$P(z) = p_0\left[\frac{Q(z)}{1 + \dfrac{1 - Q(z)}{z - 1}}\right] = p_0\left[\frac{(z-1)\,Q(z)}{z - Q(z)}\right] \qquad \ldots(23.134)$$

since $r_0 = p_0q_0$ and $R(z) = P(z)\, Q(z)$.

In order to determine p_0 it is noticed that

$$P(1) = 1 \sum_{n=0}^{\infty} p_n = \lim_{z \to 1} p_0\left[\frac{(z-1)\,Q(z)}{z - Q(z)}\right]$$

---

* Z-transform of p_n is defined by $Z(p_n) = \sum_{n=0}^{\infty} p_n z^n = P(z)$, (say).

This is also called the **probability generating function** of p_n.

**If q_n is the convolution of p_n, then $y_n - q_n{*}p_n = q_0p_n + q_1p_{n-1} + \ldots + q_{n-1}p_1 + q_np_0$.

Differentiating numerator and denominator of the right side separately to obtain

$$1 = p_0 \lim_{z \to 1} \frac{(z-1)\,Q'(z) + Q(z)}{1 - Q'(1)} = p_0 \frac{Q(1)}{1 - Q'(1)}.$$

since $Q(1) = 1$, $p_0 = 1 - Q'(1)$.

To compute $Q'(1)$ we need to determine the expression for $Q(z)$, *i.e.*

$$Q(z) = \sum_{n=0}^{\infty} q_n z^n$$

where q_n is the probability of new arrivals, and the distribution of arrivals is Poisson. Therefore,

$$Q(z) = \sum_{n=0}^{\infty} \frac{e^{-\lambda}\lambda^n}{n!} z^n = e^{-\lambda} \sum_{n=0}^{\infty} \frac{(\lambda z)^n}{n!} \text{ (since } q_n = e^{-\lambda}\lambda^n/n!\text{)}$$

$$\therefore \quad Q(z) = e^{\lambda(z-1)}. \qquad \text{...(i)}$$

Differentiating both sides of equation (i) w.r.t. 'z'.

$$Q'(z) = \lambda e^{\lambda(z-1)} \qquad \text{...(ii)}$$

Consequently, $Q'(1) = \lambda$

Thus, $p_0 = 1 - \lambda$.

From the eqn. (23.134).

$$P(z) = \frac{(1-\lambda)(z-1)\,Q(z)}{z - Q(z)} = \frac{(1-\lambda)(z-1)\,e^{\lambda(z-1)}}{z - e^{\lambda(z-1)}} = \frac{(1-z)(1-\lambda)}{1 - ze^{-\lambda(z-1)}}$$

Determination of p_n. To obtain p_n it is observed that a general expression cannot be easily obtained in this case because of the complexity of $P(z)$. However, any individual probability can be obtained by using the formula

$$p^n(0) = n!\,p_n, \text{ when } p^n(0) = \frac{\partial^n P(z)}{\partial z^n}.$$

Thus, the expected number in the system can be obtained from $P(z)$ using the formula

$$L_s = \sum_{n=0}^{\infty} np_n = P'(1) = E(n).$$

Thus, differentiating again eqn. (ii) and taking the limit as $z \to 1$, $[Q'(1) = \lambda]$

$$L_s = \frac{\lambda + Q''(1)}{2(1-\lambda)} = \frac{\lambda + \lambda^2}{2(1-\lambda)}$$

23.22. DETERMINISTIC QUEUEING MODEL

The conceptually simplest class of queueing problems are those for which probability distributions are not necessary to describe the arrival and service patterns. Instead, the units of input arrival at fixed moments of time, and service times are also fixed (constants). This type of queueing model is called the deterministic queueing model, since no probability distributions are associated with the problem in any case.

23.22-1. Model XI $(D \mid D \mid 1) : (K - 1 \mid FCFS)$

In this model

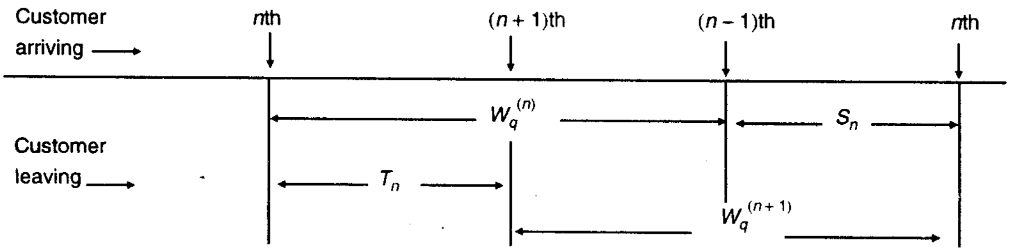
$D \to$ Deterministic arrivals, *i.e.*, inter-arrival time distribution is constant or regular

$D \to$ Deterministic service times.

Consider the elementary case of a constant rate of arrivals to a single service channel which possesses a constant sevice rate. These regularly spaced arrivals are to be served on *FCFS* basis. Let at time $t = 0$ there are no customers waiting in the queue and the channel is empty. Let λ be defined as the number of arrivals per unit time and $1/\lambda$ will then be the constant time between successive arrivals. Similarly, if μ is the service rate in terms of completion per unit time when the server is busy, then $1/\mu$ is the constant service time. Consider the case when $\lambda > \mu$, *i.e.* arrival rate is greater than the service rate, then the queue would go on increasing and will also grow beyond any bound. Each successive customer would wait longer than his predecessor for service unit, eventually customers would be waiting for ever.

To prevent this situation, forced balking is imposed on customers whenever the number in system gets increased to a certain size. This can be viewed as limiting the amount of the waiting room. The limit on the system's size will be said to be $(k-1)$ so that whenever the number of customers in the system becomes k, the entry is refused. Thus the system's size can at the most be $k-1$.

Let the number in the system at time t be $n(t)$ and the time nth arriving customer must wait in the queue to obtain service is W_q^n. Under the assumption that as soon as the service is completed another is started, the number in the system at time t is determined by the following equation.



$n(t)$ = [number of arrivals in (0, t)] —[number of services completed in (0, t)]

or $\quad n(t) = [\lambda t] - [(t - 1/\lambda)\,\mu] = [\lambda t] - [\mu t - \mu/\lambda]$,

where $[x], x \geq 0$ denotes an integer x, $n(0) = 0$.

This equation cell well be demonstrated by the following diagram.

Suppose, in particular, $1/\lambda = 3$, $1/\mu = 5$, $t = 14$,

$$n(14) = \left[\frac{14}{3}\right] - \left[\frac{14}{5} - \frac{3}{5}\right] = \left[\frac{14}{3}\right] - \left[\frac{11}{5}\right] = 4 - 2 = 2.$$

To find the waiting time. To find the waiting time in queue until the service begins, it is observed that the weights W_q^n and W_q^{n+1} of $2t$ successive customers in any single channel queue (deterministic otherwise) are related by the simple recurrence relation :

$$W_q^{(n+1)} = \begin{cases} W_q^{(n)} + s_n - T_n, & \text{if } W_q^{(n)} + s_n - T_n > 0 \\ 0 & , \text{if } W_q^{(n)} + s_n - T_n \leq 0 \end{cases}$$

where s_n is the service time of the nth customer, T_n is the inter-arrival time between the nth and $(n+1)$th customer. This can be seen by above simple diagram.

Q. 1. Write a short note on (i) deterministic queues, (ii) waiting line models. **[I.C.W.A. (June) 91]**

2. For the queueing model $D \mid D \mid 1$: $(FCFS/K-1/\infty)$, obtain the distributions for $X(t)$, the number of units in the system at time t, and for $W_q^{(n)}$, the waiting time in the queue of the nth arrival. Assume that initial the system is empty and the service time is a multiple of the inter-arrival time. **[Delhi MA/M.Sc (OR) 90]**

3. Discuss in detail a deterministic queueing model. **[Delhi MA/M.Sc (OR) 93, 92]**

23.23. QUEUEING MODELS IN SERIES

So far we have discussed the queueing models with several service stations in parallel. But, there are many real-life situations where the arriving customer passes through a successive number of service channels in order to complete his service. Such models are known as *Queues in Series* or *Queues in Tandem.* These can be further classified into following two categories.

(i) *Queues in series with possibility of queueing,* (ii) Queues in series without possibility of queueing.

In type (i) blocking is not experienced, while in type (ii) blocking is always experienced. The first type of queues in series are more practical than the second type. It is worthnoting that both the types are governed by the fact that the output of present service station (which is input to the following service station) follows the Poisson distribution.

23.23-1. Queues in Series with Possibility of Queueing

Suppose that the arrivals at service station number 1 (see Fig. 23.21) are generated from infinite population according to a Poisson distribution with mean arrival rate λ. Serviced units will move through successive service channels leaving the system at the kth service channel. The service time distribution at the ith channel is exponential with mean rate μ_i for $i = 1, 2, .., k$. In such a situation, every sub-system can be regarded as independent $(M \mid M \mid 1) : (\infty \mid FCFS)$ system, and hence the steady state probability P_n, is given (without proof) by

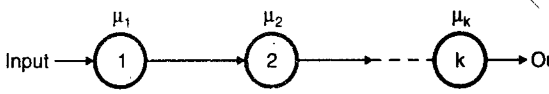


Fig. 23.21

$$P_{n_i} = (1 - \rho_i)\, \rho_i^{n_i},\ n_i = 0, 1, 2, \ldots ;$$

where $\rho_i = \lambda_i/\mu_i$ for $i = 1, 2, \ldots, k$ and n_i is the number of customers in ith sub-system.

By the property of independent probabilities, we have

$$P(n_1, n_2, \ldots, n_k) = \prod_{i=1}^{k} P_{n_i} = \prod_{i=1}^{k} (1 - \rho_i)\, \rho_i^{n_i}$$

23.23–2. Queues in Series without Possibility of Queueing

In this situation the customers are served at each of two stages arranged in series. The arrivals follow Poisson law with parameter λ. Thus, each stage is considered to have single server, and the service time in ith stage being exponentially distributed with parameters μ_i, $i = 1, 2, \ldots$ Here it is assumed that no queue is permitted in front of each stage. This causes a phenomenon of blocking. By blocking we mean the situation that takes place when a unit has completed service in the first stage but cannot proceed because the second stage is not complete, consequently, arriving customers turn away without service because of the first stage is blocked. The probability of such event is called the *rate of loss-call* which is considered to be an important measure of efficiency for the system without queues.

To find the rate of loss-call :

In order to find the *rate of loss-call* for a *two-stage tandem queueing system* the arrival rate is $\lambda > 0$ and the service rate of *first* and *second* stage are $\mu_1 > 0$, and $\mu_2 > 0$, respectively.

Now, the following steady-state probabilities are defined :

$p(0, 0)$ = Prob. *(that both stages are empty)*
$p(0, 1)$ = Prob. *(that first is empty and second is full)*
$p(1, 0)$ = Prob. *(that first stage is full and the second is empty)*
$p(1, 1)$ = Prob. *(that both the stages are full, first is working)*
$p(b, 1)$ = Prob. *(that first is blocked and the second is full).*

Thus the system of governing steady state equations are :

$$\lambda p(0, 0) = \mu_2 p(0, 1)$$
$$(\lambda + \mu_2)\, p(0, 1) = \mu_1 p(1, 0) + \mu_2 p(b, 1)$$
$$\mu_1 p(1, 0) = \mu_2 p(1, 1) + \lambda p(0, 0)$$
$$(\mu_1 + \mu_2)\, p(1, 1) = \lambda p(0, 1)$$
$$\mu_2 p(b, 1) = \mu_1 p(1, 1)$$
$$p(0, 0) + p(0, 1) + p(1, 0) + p(1, 1) + p(b, 1) = 1$$

The solution of these equations is given by

$$p(0, 0) = \lambda^2 \mu_1 (\mu_1 + \mu_2)/\delta$$
$$p(0, 1) = \lambda \mu_1 \mu_2 (\mu_1 + \mu_2)/\delta$$
$$p(1, 0) = \lambda \mu_2^2 (\lambda + \mu_1 + \mu_2)/\delta$$
$$p(1, 1) = \lambda^2 \mu_1 \mu_2/\delta$$
$$p(b, 1) = \lambda^2 \mu_1^2/\delta.$$

where, for convenience, $\delta = \mu_1 (\mu_1 + \mu_2)(\lambda^2 + \lambda\mu_2 + \mu_2^2) + \lambda (\mu_1 + \mu_2 + \lambda)\, \mu_2^2$

Hence the *rate of loss-call* is given by $L = p(1, 0) + p(1, 1) + p(0, 1)$.

23.24. QUEUEING CONTROL

During past three decades there has been a rapidly increasing interest in the study of designing and controlling of behaviour of queueing system. Majority of queueing literature involve *prescriptive* models rather than *descriptive*. Prescriptive models are viewed as static optimization models.

Static optimization models are those in which steady-state conditions are set up for the system and some long run average criterion (such as cost and/or profit) is determined.

In static models, the configuration of the system is set once for all.

If the queueing systems depend upon time and are controlled, then these systems are known as *dynamic control systems.* Some optimization models are the mixture of *static* and *dynamic* categories. But, if the state dependent system is controlled, then it comes under *dynamic control.* Much amount of work has been done on dynamic control systems.

Objectives of Dynamic control :

Following are the objectives of dynamic control :

(*a*) ***Arrival Process control***

(*i*) To accept or reject the control
(*ii*) To adjust mean arrival rate
(*iii*) Customer exercised control
(*iv*) Self versus social optimization
(*v*) Projection times.

(*b*) ***Service Process control***

(*i*) Varying the number of servers
(*ii*) Varying the service times.

Control of Queue Discipline :

There is one more branch of optimization which is named as *'control of Queue Disciplines'*, Priority models, scheduling models, and allocation of customers to multi-server fall in this category.

Q. 1. Discuss the queueing model which applies to a queueing system having a single service channel, Poisson input, exponential service, assuming that there is no limit on the system capacity while the customers are served on a first come basis out basis.

2. Give essential characteristics of the queueing process. What are non-poission queues ? **[Meerut (Stat.) 95]**

3. "Queueing theory can be used effectively in determining optimal service levels." Elucidate this statement with the help of an example. **[Delhi (M.B.A.) Dec. 94]**

4. Define Queue. Write the characteristics of Queueing system. **[Bhubnashwar (IT) 2004]**

SELF-EXAMINATION PROBLEMS

1. At a one-man barber shop, customers arrive according to Poisson distribution with a mean arrival rate of 5 per hour and his hair cutting time was exponentially distributed with an average hair cut taking 10 minutes. It is assumed that because of his excellent reputation, customers were always willing to wait. Calculate the following.
(i) Average number of customers in the shop and the average number of customers waiting for a hair cut.
(ii) The percentage of customers who have to wait prior of getting into the barber's chair.
(iii) The per cent of time an arrival can walk without having to wait.
[Hint. Here $\lambda = 1/12$ per minute, $\mu = 1/10$ per minute.]
[Ans. (i) $L_s = 4{\cdot}8$, $L_q = 4$ (approximately)
(ii) P(queue size ≥ 1) = $\rho = 0.833$. Therefore, percentage of customers who have to wait = 83.3%
(iii) Per cent of time an arrival can walk without waiting = 100 – 83.3 = 16.7%.]

2. An overhead crane moves jobs from one machine to another and must be used every time a machine requires loading or unloading. The demand for service is random, data taken by recording the elapsed time between service calls followed as exponential distribution having a mean of a call every 30 minutes. In a similar manner, the actual service time of loading or unloading took an average of 10 minutes. If the machine time is valued at Rs. 8.50 per hour, how much does the down time cost per day ? (Assume one day = 8 working hrs.)
[Hint. $\lambda = 60/30$ per hour, $\mu = 60/10$ per hour. Find Down time (W_s) = 0.25 per hour. Since daily demand = 8λ = 16 calls per day, and each call requires 0.25 hour,
$\therefore$ Total cost per day = Rs. $(16 \times 0.25 \times 8.50)$.]

3. A duplicating machine maintained for office-use is used and operated by people in office who need to make copies, mostly secretaries. Since the work to be copied varies in length (number of pages of the original) and copies required, the service rate is randomly distributed, but it does approximate a Poisson having a mean service rate of 10 jobs per hour. Generally, the requirements for use are random over the entire 8-hour work day but arrive at a rate of 5 per hour. Several people have noted that a waiting line develops occasionally and have questioned the policy of maintaining only one unit. If the time of a secretary is valued at Rs. 3.50 per hour, then determine :
(a) The per cent of time the equipment is used. (b) The per cent of time that an arrival has to wait.
(c) Average waiting time of an arrival in the system.
(d) The average cost due to waiting and operating the machine. **[I.C.W.A. (June) 90]**

[**Hint.** Here $\lambda = 5$, $\mu = 10$.]

(a) Utilization factor (ρ) = $\lambda/\mu = 1/2$. Therefore, equipment is used 50% of the time. (b) Busy period = 0.50.

(c) $W_s = \frac{1}{\mu - \lambda} = 0.20$ hr.

(d) Since average cost per job = $W_s \times$ (Rs. 3.50) = Re. 0.70, cost per day = 8 × 5 × Re. 0.70 = Rs. 28 per day.]

4. Consider a single server queueing system with a Poisson input, exponential service times. Suppose the mean arrival rate is 3 calling units per hour, the expected service time is 0.25 hour, and the maximum permissible number of calling units in the system is two. Derive the steady state probability distribution of the number of calling units in the system, and the, calculate the expected number in the system.

Derive the formula $P_n = \frac{(1-\rho)\,\rho^n}{1-(\rho)^{N+1}}$ $(\rho \neq 1)$

[**Hint.** Here $\lambda = 3$, $1/\mu = 25/100$, $N = 2$.

$$P_n = \frac{(0.25)\,(0.75)^n}{1-(0.75)^3} = (0.43)\,(0.75)^n,\ P_0 = \frac{1-\rho}{1-\rho^{2+1}} = \frac{0.25}{1-(0.75)^3} = 0.43.$$

$$L_s = \sum_{n=0}^{N} nP_n = \sum_{n=0}^{N} n\,(\rho^n)\,P_0 = P_0\,(\rho + 2\rho^2) = 0.81.]$$

5. At a railway station, only one train is handled at a time. The railway yard is sufficient only for 3 trains to wait while other is given signal to leave the station. Trains arrive at the station at an average rate of 6 per hour and the railway station can handle them on an average of 12 per hour. Assuming Poisson arrivals and exponential service distribution, find the steady state probabilities for the various number of trains in the system. Also find the average waiting time of a new train coming into the yard.

[**Hint.** $\lambda = 6$, $\mu = 12$, $N = 3$. Find, $P_0 = 0.53$.

Since $P_n = P_0\,(\lambda/\mu)^n$, find $P = 0.27$, $P_2 = 0.13$, $P_3 = 0.07$.

Therefore $L_q = 1\,(0.27) + 2\,(0.13) + 3\,(0.07) = 0.74$.

Thus the average number of trains in the queue is 0.74 and each train takes on an average 1/2 hour for getting service. Therefore, $W_q = (0.74)\,(1/12)$ hour = 3.8 minutes.]

6. A telephone company is planning to install telephone booths in a new airport. It has established the policy that a person should not have to wait more than 10 per cent of the times he tries to use a phone. The demand for use is estimated to be Poisson with an average of 30 per hour. The average phone call has an exponential distribution with a mean time of 5 minutes. How many phone booths should be installed.

[**Hint.** Here $\lambda = 30$ per hour, $\mu = 60/5$ per hour, $\lambda/\mu = 2.5$.

Assume that the company decides to install two telephone booths, then $2\lambda = 24$ which is less than λ the arrival rate. So the telephone company must have at least three telephone booths to meet the demand for service.

Now consider $s = 4$ for this to obtain $P_6 = 0.13$. Next consider $s = 6$. Then $P_6 = 0.047$.

Thus an installation of six phones would give a probability of 0.047 that a customer would have to wait. Since this is less than 0.13 it is the smallest number that will meet the company policy.]

7. An oil company is constructing a service station on a high way. Traffic analysis indicates that customers arrivals over most of the day would approximate a Poisson distribution with a mean of 10 automobiles per hour, with the service time distribution approximating the negative exponential. If 4 pumps are installed,

(a) what is the probability that an arrival would have to wait in line ?

(b) find out the average waiting time, average time spent in the system and the average number of automobiles in the system ?

(c) for what percentage of time would a pump be idle on an average ?

[**Hint.** Here $\lambda = 30$, $\mu = 10$, $s = 4$, compute $P_0 = 0.0377$

$$P_1 = \left(\frac{\lambda}{\mu}\right) P_0 = 0.1131,\ P_2 = \frac{1}{2\,!}\left(\frac{\lambda}{\mu}\right)^2 P_0 = 0.1697,\ \ P_3 = \frac{1}{3\,!}\left(\frac{\lambda}{\mu}\right)^3 P_0 = 0.1697,\ P_4 = \frac{1}{4\,!}\left(\frac{\lambda}{\mu}\right)^4 P_0 = 1.1272.$$

(a) Prob. (an arrival have to wait) = $1 - P_0 - P_1 - P_2 - P_4 = 0.3826$ = 62% nearly

(b) $L_s = \frac{\lambda\mu\,(\lambda/\mu)^s\,P_0}{(s-1)\,(s\mu-\lambda)^2} + \frac{\lambda}{\mu} = 1.53 + 3 = 4.53$ automobiles.

$W_q = \frac{L_q}{\lambda} = \frac{1.53}{30} = 0.509$ hour or 3.05 minutes, $W_s = W_q + 1/\mu = 0.0509 + 0.1 = 0.1509$

(c) Number of idle pumps :	4	3	2	1	0
Probability :	0.0377	0.1131	0.1697	0.1697	0.1272
	$(=p_0)$	$(=p_1)$	$(=p_2)$	$(=p_3)$	$(=p_4)$

Average number of idle pumps = 0.9992 , Prob. (any pump will be idle) = $\frac{0.9992}{4}$ = 24.98%.]

8. A group of engineers has two terminals available to aid in their calculations. The average computing job requires 20 minutes of terminal time, and each engineer requires some computation about once every 0.6 hours, *i.e.*, the mean time between a call for service is 0.5 hour. Assume these are distributed according to an exponential distribution. If there are six engineeers in the group, find :

(a) The expected number of engineers waiting to use one of the terminals.

(b) The total lost time per day.

[**Hint.** Here $K = 6$, $R = 2$, $\lambda = 2$, $\mu = 3$, compute $P_0 = 0.0268$, $P_2 = 0.18$, $P_3 = 0.24$, $Pusb4 = 0.24$, $P_5 = 0.16$, $P_6 = 0.05$.

(a) $L_q = \sum_{n=R}^{k} (n - R)\, P_n = 0 + 1P_3 + 2P_4 + 3P_5 + 4P_6 = 1.40$

(b) Time lost per day = 8 × 1.40 = 11.2 hours per day.]

9. A company used two large machines in a processing operation. Each machine is running all the time, except when down for repairs. Given that a machine is running at time t, the probability that it fails between t and $t + dt$ is λdt with λ in terms of failure per hour. Any time a machine fails it is immediately assigned a repair crew (assume at least 2 are always available), for each machine the repair time is negative exponential with average $1/\mu$ hours.

(a) Find the steady-state probabilities P_0, P_1 and P_2.

(b) What is the average down time of a machine ?

(c) Let $\lambda = 1/2$ and down time costs Rs. 25μ per hour to operate the repair service at the rate of μ. Find out the optimal service rate.

[**Hint.** Let n be the number of mahcines down for repair at any given time. Here λ and μ depend on n.

(a) $P_1 = \dfrac{2\lambda}{\mu} P_0$ and $P_2 = \dfrac{\lambda^2}{\mu^2} P_0$, compute $P_0 = \dfrac{\mu^2}{(\lambda + \mu^2)}$.

Therefore, $P_1 = \dfrac{2\lambda\mu}{(\lambda + \mu)^2}$, $P_2 = \dfrac{\mu^2}{(\lambda + \mu)^2}$

(b) $L_s = \sum_{n=0}^{2} n\, P_n = P_1 + 2P_2 = \dfrac{2\lambda}{\lambda + \mu}$

Hence average down time of a machine $= \dfrac{2\lambda/(\lambda + \mu)}{2} = \dfrac{\lambda}{\lambda + \mu}$ as a fraction

(c) Total cost per hour $= 100 \times \dfrac{\lambda}{\lambda + \mu} + 25\mu = \dfrac{100}{1 + 2\mu} + 25\mu$ (when $\lambda = 1/2$)

This is minimized when $\mu = 0.915$. So optimal service rate $1/\mu = 1.09$ hours].

10. Two types have identical jobs. Each type letter is dictated by a manager. Suppose that letters to be typed arrive at random (following Poisson distribution) at a rate of three per hour for each typist. Suppose that each typist can type four letters per hour on the average (following exponential distribution) :

(i) Assuming that each typist does here own work, what is the expected waiting time for a letter (time before work is started on a letter)?

(ii) Suppose that the two typists are 'pooled'. That is, letters are sent to the two together and are done by whoever is free, in the order of arrival. What is the expected waiting time for a letter under this arrangement ?

[**Hint.** (i) Here $\lambda = 3/60$, $\mu = 4/60$, $s = 2$. Use formula for W_q of Model IV ($M \mid M \mid s$)

(ii) Here $\lambda = 3/60$ (unchanged), $\mu = (4 + 4)/60$. Use formula for W_q of Model I ($M \mid M \mid 1$)]

[**Ans.** (i) 27/11 minutes, (ii) 4.5 minutes.]

11. The manager of a Capital Finance loan office was to make a decision regarding its service rate on new loans. One service rate is 0.5 unit per hour with a service time variance of 3 hours and another is a service rate of 0.4 unit per hour with a service time variance of 2 hours. The difference in rate per hour represents explaining to the customer the other services sometimes in the future. The cost of waiting per customer for an hour is estimated to be Rs. 3.00 while the cost of servicing one arrival is Rs. 1.50. The number of arrivals has a Pisson distribution with a mean of 0.3 units per hour. What should the manager do ?

12. A certain queueing system has a Poisson input with a mean arrival rate of two calling units per hour. The service-time distribution is exponential with a mean of 0.4 hour. The marginal cost of providing each server is Rs. 4 per hour, where it is estimated that the cost which is incurred by having each calling unit idle (*i.e.*, in the queueing system) is Rs. 100 per hour. Determine the number of servers that should be assigned to the system in order to minimize the expected total cost per hour.

13. At point, let there be six unloading berths and four unloading crews. When all the berths are full, arriving ships are diverted to an overflow facility 20 miles down the river. Tankers arrive according to a Poisson process with a mean of one every 2 hour. It takes an unloading crew, on the average ten hours to unload a tanker, the unloading time following an exponential distribution. Find :

(i) On the average, how many tankers are at the port ?

(ii) On the average how long does a tanker spend at the port ?

(iii) What is the average arrival rate at the overflow facility ?

14. A repairman is to be hired to repair machines which breakdown at an average rate of three per hour. Breakdowns are distributed in time in a manner that may be regarded as Poisson. Non-productive time on any one machine is considered to cost the company Rs. 5 per hour. The company has narrowed the choice down to two repairman one slow and cheap, the other fast but expensive. The slow cheap repairman demands Rs. 3 per hour, in return he will service the broken down machines exponentially at an average rate of four per hour. The fast expensive repairman demands Rs. 5 an hour, and will repair machines exponentially at an average rate of six per hour. Which repairman should be hired.

15. Consider a single station queueing system with a Poisson input with known mean arrival rate. Suppose that the service time distribution is unknown but that the expected service time $1/\mu$ is known. Compare the expected waiting time in the queue if the service time distributions were (i) exponential (ii) constant.

16. For the case of two channels, Poisson arrivals and exponential service, show the following :

(i) Probability that both the channels are empty is $\frac{2\mu - \lambda}{2\mu + \lambda}$.

(ii) Expected number in the system is $\frac{4\lambda\mu}{2\mu^2 - \lambda^3}$.

17. A department store has a single cashier. During the rush hours, customers arrive at a rate of 20 customers per hour. The average number of customers that can be processed by the cashier is 24 per hour. The average number of customers that can be processed by the cashier is 24 per hour. Assume that the conditions for use of the single-channel queueing model apply :

(a) What is the probability that the cashier is idle ?
(b) What is the average number of customers in the queueing system ?
(c) What is the average time a customer spends in the system ?
(d) What is the average number of customers in the queue ?
(e) What is the average time a customer spends in the queue waiting for services ?

18. A single channel queueing system has Poisson arrivals and exponential service times. The mean arrival rate is 88 transactions per hour and the mean service rate is 23 per hour. Determine :

(a) The average time a customer will wait in the system.
(b) The average number of customers waiting in the queue.
(c) The utilization factor of the system.

19. Workers come to a tool store room to enquire about the special tools (required by them) for a particular job. The average time between the arrivals is 60 seconds and the arrivals are assumed to be Poisson distribution. The average service time is 40 seconds. Determine :

(a) average queue length,
(b) average length of non-empty queue,
(c) average number of workers in the system including the workers being attended,
(d) mean waiting time of an arrival,
(e) average waiting time of an arrival (workers) who waits. **[Virbhadra 2000]**

[**Ans.** (a) 1.33 workers, (b) 3 workers, (c) 2 workers, (d) 1.33 minutes per worker, (e) 2 minutes per worker]

20. Problems arrive at a computing centre in Poisson fashion with a mean arrival rate of 25 per hour. The average computing job requires 2 minutes of terminal time. Calculate the following :

(a) Average number of problems waiting for the computer use.
(b) The per cent of times an arrival can walk right in without having to wait.

21. In a bank cheques are cashed at a single 'teller' counter. Customers arrive at the counter in a Poisson manner at an average rate of 30 customers per hour. The teller takes on an average a minute and a half to cash a cheque. The service time has been shown to be exponentially distributed.

(i) Calculate the percentage of time the teller is busy.
(ii) Calcualte the average time a customer is expected to wait.

[**Ans.** (i) 75%, (ii) 6 minutes.]

22. In a Tool Crib manned by a single Assistant the operators arrive at the tool crib at the rate of 10 per hour. Each operator needs 3 minutes on an average to be served. Find out the loss of production due to waiting of an operator in a shift of 8 hours if the rate of production is 100 units per shift.

23. In a bank with a single server, there are two chairs for waiting customers. On an average one customer arrives every 10 minutes and each customer takes 5 minutes for getting served. Making suitable assumptions, find :

(i) the probability that an arrival will get a chair to set down,
(ii) the probability that an arrival will have to stand, and
(iii) expected waiting time of a customer.

[**Ans.** (i) 7/8, (ii) 1/8, (iii) 5 minutes.]

24. Trucks arrive at a factory for collecting finished goods for transportation to distant markets. As and when they come they are required to join a waiting line and are served on first come first seved basis. Trucks arrive at the rate of 10 per hour whereas the loading rate is 15 per hour. It is also given that arrivals are Poisson and loading is exponentially distributed. Transporters have complained that their trucks have to wait for nearly 12 hours at the plant. Examine whether the complaint is justified. Also, determine probability that the loaders are idle in the above problem.

25. In a service department manned by one server, on an average one customer arrives every 10 minutes. It has been found that each customer requires 6 minutes to be served. Find out :

(i) Average queue length, (ii) Average time spent in the system,

(iii) The probability that there would be two customers in the queue.
[**Ans.** (i) 0.9 customers, (ii) 15 minutes, (iii) 1.44%]

26. At Dr. Parachi's clinic, patients arrive at an average of 6 patients per hour. The clinic is attended to by Dr. Prachi herself. Some patients require only the repeat prescription, some come for minor check-up while some others require thorough inspection for the diagnosis. This takes the doctor six minutes per patient on the average. It can be assumed that arrivals follows a Poisson distribution and the doctor's inspecion time follows an exponential distribution. Determine :
(i) The per cent of times a patient can walk right inside the doctors cabin, without having to wait;
(ii) the average number of patient in Dr. Prachi's clinic;
(iii) the average numeber of patients waiting for their term, and
(iv) the average time a patient spends in the clinic.
[**Ans.** (i) 40%, (ii) 1½ patient, (iii) 0.90 patients, (iv) 15 minutes]

27. Customers arrive at a sales-counter maned by a single person according to a Poisson process with a mean rate of 20 per hour. The time required to serve a customer has an exponential distribution with a mean of 100 seconds. Find the average waiting time of a customer.
[**Ans.** (i) $\frac{20}{36\,(36-20)} \times 3600$ seconds, (ii) $\frac{1}{36-20} \times 3600$ seconds]

28. A post-office has 3 windows providing the same services. It receives on average 30 customers/hr. Arrivals are Poisson distributed and service time exponenntially distributed. The post office serve on average 12 customers per hour.
(i) What is the probability that a customer will be served immediately ?
(ii) What is the probability that a customer will have to wait ?
(iii) What is the average total time that a customer must spend in the post-office ?

29. A fertilizer company distributes its products by trucks loaded at its only loading station. Both, company trucks and contractors' trucks are used for this purpose. It was found out that on an average every 5 minutes one truck arrived and the average loading time was 3 minutes. 40 per cent of the trucks belong to the contractors. Making suitable assumption determine :
(i) The probability that a truck has to wait. (ii) The waiting time of a truck that waits.
(iii) The expected waiting time of contractors' trucks per day.
[**Ans.** (i) 3/5, (ii) 1/8, (iii) $12 \times 12 \times \frac{40}{100} \times \frac{12}{20\,(20-22)}$]

30. Telephone users arrive at a booth following a Poisson distribution with an average time of 5 minutes one arrival and the next. The time taken for a telephone call is on an average 3 minutes and it follows an exponential distribution. What is the probability that the booth is busy ? How many more booths should be established to reduce the waiting time to less than or equal to half of the present waiting time ?

31. Arrivals of customers to a payment counter (only one) in a bank follow Poisson distribution with an average of 10 per hour. The service time follows negative exponetial distribution with an average of 4 minutes.
(i) What is the average number of customers in the queue ?
(ii) The bank will open one more counter when the waiting time of a customer is at least 10 minutes. By how much the flow of arrivals should increase in order to justify the second counter.

32. In a store customers arrive in a Poisson stream with mean 60 per hour. The service time is exponential with mean of 0.005 hours. How many clerks should be available if the expected waiting time in the system should be less than 10 minutes ?

33. The average rate of arrivals at a service store is 30 per hour. At present there is one cashier who on average attends to 45 customers per hour. The store proprietor estimates that each extra minute of system process time per customer means a loss of Rs. 0.50. An assistant can be provided to the cashier and in that case the service unit can deal with 75 customers per hour. The wage rate of the assistant is Rs. 15 per hour.
Is it worth employing an assistant ?

34. In a factory, the machine breakdown on an average rate is 10 machines per hour. The idle time cost of a machine is estimated to be Rs. 20 per hour. The factory works 8 hour a day. The factory manager is considering 2 mechanics for repairing the machines. The first Mechnic A takes, about 5 minutes on an average to repair a machine and demand wages Rs. 10 per hour. The second mechanic B takes about 4 minutes in repairing a machine and demand wages at the rate of Rs. 15 per hour. Assuming that the rate of machine breakdown is Poisson distributed and the repair rate is exponentially distributed, which of the two mechanics should be engaged ? **[Delhi (M.Com.) 90]**

35. Assume that two different tools C and D can be leased for use in repairing machines at the rate of Rs. 35 and Rs. 10 per hour. The rate of break-down necessitating repair is Poisson distribution with a mean of 3 per hour for the period of time the machine is out of service. The tools repair machines at an exponentially distributed rate having a mean of 4 per hour for a tool C and 6 per hour for the tool D. Which of the two tools should be leased by the company.

36. A warehouse has only one loading dock manned by a three person crew. Trucks arrive at the loading dock at an average rate of 4 trucks per hour and the arrival rate is Poisson distributed. The loading of a truck takes 10 minutes on an average and can be assumed to be exponentially distributed. The operating cost of a track is Rs. 20 per hour and the members of the loading crew and paid @ Rs. 6 each per hour. Would you advise the truck owner to add another crew of three persons ?
[**Ans.** (a) 0.83, (b) 8.3%]
[**Hint.** Total hourly cost = loading crew cost × cost of waiting :
With present crew :

Loading cost = no. of loaders × hourly wage rate = Rs. 18/hr.

$$\text{Waiting time cost} = \begin{pmatrix}\text{expected waiting time}\\ \text{per truck } (W_s)\end{pmatrix} \times \begin{pmatrix}\text{expected arrivals}\\ \text{per hour } (\lambda)\end{pmatrix} \times \begin{pmatrix}\text{hourly}\\ \text{waiting cost}\end{pmatrix}$$

= Rs. 40/hr.

∴ Total cost = Rs. 18 + Rs. 40 = Rs. 58/hr.

After proposed crew addition :

$$\text{Total cost} = 6 \times 6 + \frac{4}{12-4} \times 20 = \text{Rs. } 46/\text{hr.}$$

On comparison, it will be *economical* to add a *crew of 3 loaders*.]

37. A firm has several machines and wants to install its own service facility for the repair of its machines. The average breakdown rate of the machines is 3 per day. The repair time has exponential distribution. The loss incurred due to the lost time of an inoperative machine is Rs. 40 per day. There are two repair facilities available. Facility A has an installation cost of Rs. 20,000 and the facility B cost Rs. 40,000. With facility A, the total labour cost is Rs. 5,000 per year and with facility B the total labour cost is Rs. 5,000 per year and with facility B the total cost is Rs. 8,000 per year. Facility A can repair 4½ machines per day and the facility B can repair 5 machines per day. Both facilities have a life of 4 years. Which facility should be installed.
[**Ans.** Facility A]

38. In the machine shop of XYZ company four overhead cranes serve a number of production machines. If all cranes are busy and one machinist must wait for service, the waiting cost is Rs. 4.50 per hour (wage rate, fringe benefits and non-productive costs). On the other hand, the overhead cost of the crane is Rs. 5.80 per hour (wage rate, fringe benefits and other costs). Empirical data gathered indicates that the number of machinists calling for crane service follow the Poisson distribution with an average rate of 5 calls per hour. The average service time is exponentially distributed with a time of twenty minutes per call. The crane services the machines on the first come first served basis. The foreman incharge of manufacturing wants to know how many cranes are needed to keep the machinists cost of waiting time and the overhead cost of the cranes at a minimum. Use an 8-hour day in your calculations.

39. A steel fabrication plant was considering the installation of a second tool crib in the plant to save walking time oı the skilled craftsmen who checkout equipment at the tool cribs. The Poisson/exponential assumptions about arrivals are justified in this case. The time of the craftsmen is valued at Rs. 20/hr. The current facility receives an average of 10 calls per hour; with two cribs, each would average five calls per hour. Currently, there are two attendants, each of whom services one craftsman at a time; each has a service rate of eight carftsmen per hour. Each could do just as well in a separate tool cribs. There would be added average inventory costs over the year of Rs. 2/hour with the separate tool cribs, however each craftsmen would require six minutes less walking time per call. Evaluate the proposal to set up a new crib so that each attendant would run one crib.

40. A company has two manufacturing shops and two tool cribs, one for each shop. Both the tool cribs handle almost identical tools, gauges and measuring instruments. Analysis of service time shows a negative exponential distribution with a mean of 2.5 minutes per workman. Arrivals of workmen follow Poisson distribution with a mean of 18 per hour. The production manager feels that if tool cribs are combined for both shops, efficiency will improve and waiting time in the queue will reduce. Do you agree with his opinion ?
[**Ans.** (i) W_q = 7.5 minutes, (ii) for $s = 2$, W_q = 45/14 minutes]

41. Customers arrive at the rate of 20 per hour and the present serving arrangements can cope with 30 per hour for an eight-hour day. Find—
(a) the average time in the queue
(b) the implied value of customer's time, if the owner of the service has considered but rejected a faster service arrangement which would cost an extra Rs. 20 for an eight hour day and would raise the service rate to 40 per hour.
The following formulae are given :

$$\text{Average time in system} : \frac{1}{1-\rho} \times \frac{1}{\mu}, \text{ Average time in queue} : \frac{\rho}{1-\rho} \times \frac{1}{\mu}$$

[**Hint.** (a) Here $\lambda = 20$/hr., $\mu = 30$/hr., *i.e.*, $\rho = 2/3$. **Ans.** 4 minutes.
(b) With a service rate of 40 per hour : $\rho = 20/40 = 0.5$.

$$\text{Old average time in system} = \frac{1}{1-0.67} \times \frac{1}{30} \times 60 = 6 \text{ minutes.}$$

$$\text{New average time in system} = \frac{1}{1-0.5} \times \frac{1}{30} \times 60 = 3 \text{ minutes.}$$

Assuming break-even value of customers' waiting time is Rs. k per hour, we have

$$8\left(\frac{6}{60} - \frac{3}{60}\right) \cdot 20 \cdot k < 20 \quad \text{or} \quad k < 2.50 \text{ per hour.]}$$

42. The repair of a Lathe requires four steps to be completed one after another in a certain order. The time taken to perform each step follows exponential distribution with a mean of 15 minutes and is independent of other steps. Mechine breakdown follows Poisson process with mean rate of 5 break-downs per hour. Answer the following :
(i) What is the expected idle time of the machine, assuming there is only one repair-man available in the workshop ?
(ii) What is the average waiting time of a breakdown machine in the queue ?
(iii) What is the expected number of broken-down machines in the queue ?
[**Hint.** Here $\lambda = 5$ per hour, $\mu = 12$ per hour, and $k = 4$ steps. Use formula for L_s , L_q and W_s.]

43. At a certain airport flight arrivals can be taken to follow a poisson distribution inspite of scheduples due to many constraints and uncertainties. The mean time between two arrivals is 12 minutes. There are two separate runways, one for lending and one for take off. The airport service time for landing flights is exponentially distributed with a mean of 8 minutes per flight. When one flight is being serviced another incoming flight cannot be cleared for landing due to safety regulations.
   (i) What is the chance that an incoming flight is straight away cleared for landing.
   (ii) What is the average waiting time for an incoming flight to be cleared for landing.
   (iii) What is the probability that the waiting time is in excess of 50% of the average waiting time found in (ii) ?
   (iv) There is a proposal that the airport should be converted into an international terminal. One of the important requirements of an international terminal is that the average waiting time should be reduced. If an additional runway is provided with independent service facility for landing, of a nature similar to the existing one, what will be the average waiting time of an incoming flight to be cleared for landing ? **[Karnataka BE (CSE) 94]**

44. In a bank with a single server, there are two chairs for waiting customers. On an average one customer arrives every 10 minutes and each customer takes 5 minutes for getting served. Making suitable assumptions, find out :
   (i) the probability that an arrival will get a chair to sit down.
   (ii) the probability that an arrival will have to stand :
   (iii) expected waiting time of a customer; and
   (iv) the length of a non-empty queue. **[Delhi (FMCI) 2000]**

45. What is queueing theory ? Describe the different types of costs involved in a queueing system. In what areas of management can queueing theory be applied successfully ? Give examples. **[JNTU (MCA III) 2004]**

46. Describe a single server waiting line model. Give an example of real life for each of the following queueing models :
   (i) First come – First Served
   (ii) Last come – First served
   (iii) Random pick service
   (iv) Customers stay only it served instanlly. **[JNTU (MCA III) 2004]**

OBJECTIVE QUESTIONS

1. Customer behaviour in which be moves from one queue to another in a multiple channel situation is
   (a) balking. (b) reneging. (c) jockeying. (d) alternating.
2. Which of the following characteristics apply to queueing system
   (a) Customer population. (b) Arrival process. (c) Both (a) and (b). (d) Neither (a) nor (b).
3. Which of the following is not a key operating characteristic for a queuing system?
   (a) Utilization factor. (b) Per cent idle time.
   (c) Average time spent for waiting in system and queue. (d) None of the above.
4. Priority queue discipline may be classified as
   (a) finite or infinite. (b) limited and unlimited.
   (c) pre-emptive or non-pre-emptive. (d) all of the above.
5. Which symbol describes the inter-arrival time distribution ?
   (a) *D*. (b) *M*. (c) *G*. (d) All of the above.
6. Which of the following relationships is not true
   (a) $W_s = W_q + 1/\mu$. (b) $L_s = \lambda\, W_s$. (c) $L_s = L_q + 1/\lambda$. (d) $L_q = \lambda\, W_q$.
7. The calling population is assumed to be infinite when
   (a) arrivals are independent of each other. (b) capacity of the system is infinite.
   (c) service rate is faster than the arrival rate. (d) all of the above.
8. Which of the cost estimates and performance measures are not used for economic analysis of a queuing system?
   (a) Cost per server per unit of time. (b) Cost per unit of time for a customer waiting in the system.
   (c) Average number of customers in the system. (d) Average waiting time of customers in the system.
9. A calling population is considered to be infinite when
   (a) all customers arrive atonce. (b) arrivals are independent of each other.
   (c) arrivals are dependent upon each other. (d) all of the above.
10. The cost of providing service in a queueing system decrease with
   (a) decreaed average waiting time in the queue. (b) decreased arrival rate.
   (c) increased arrival rate. (d) none of the above.

Answers

1. (c) **2.** (c) **3.** (d) **4.** (c) **5.** (d) **6.** (c) **7.** (a) **8.** (d) **9.** (b) **10.** (d).

❋❋❋

24

JOB SEQUENCING

24.1. INTRODUCTION

Suppose there are n jobs to perform, each of which requires processing on some or all of m different machines. The effectiveness (*i.e.* cost, time or mileage, etc.) can be measured for any given sequence of jobs at each machine, and the most suitable sequence is to be selected (which optimizes the effectiveness measure) among all $(n\,!)^m$ theoretically possible sequences. Although, theoretically, it is always possible to select the best sequence by testing each one, but it is practically impossible because of large number of computations.

In particular, if $m = 5$ and $n = 5$, the total number of possible sequences will be $(5\,!)^5 = 25{,}000{,}000{,}000$. Hence the effectiveness for each of $(5\,!)^5$ sequences is to be computed before selecting the most suitable one. But, this approach is practically impossible to adopt. So easier methods of dealing with such problems are needed.

Before proceeding to our actual discussion we should explain what the sequencing problem is.

The problem of sequencing may be defined as follows :

Definition. Suppose there are n jobs (1, 2, 3, ..., n), each of which has to be processed one at a time at each of m machines $A, B, C, \ldots$ The order of processing each job through machines is given (for example, job 1 is processed through machines A, C, B—in this order). The time that each job must require on each machine is known. The problem is to find a sequence among $(n\,!)^m$ number of all possible sequences (or combinations) (or order) for processing the jobs so that the total elapsed time for all the jobs will be minimum.

Mathematically, let

$A_i =$ time for job i on machine A,

$B_i =$ time for job i on machine B, etc.

$T =$ time from start of first job to completion of the last job.

Then, the problem is to determine for each machine a sequence of jobs $i_1, i_2, i_3, \ldots, i_n$, where $(i_1, i_2, i_3, \ldots, i_n)$ is the permutation of the integers which will minimize T.

Q. 1. Define the problem of Sequencing. **[Agra 99]**

2. Explain what do you mean by a sequencing problem. **[JNTU (B. Tech.) 2003; Meerut 2002]**

24.2. TERMINOLOGY AND NOTATIONS

The following *terminology* and notations will be used in this chapter.

(1) **Number of Machines.** It means the service facilities through which a job must pass before it is completed.

For example, a book to be published has to be processed through composing, printing, binding, etc. In this example, the book constitutes the *job* and the different processes constitute the *number of machines.*

(2) **Processing Order.** It refers to the order in which various machines are required for completing the job.

(3) **Processing Time.** It means the time required by each job on each machine. The notation T_{ij} will denote the processing time required for ith job by jth machine ($i = 1, 2, \ldots, n; j = 1, 2, \ldots, m$).

(4) **Idle Time on a Machine.** This is the time for which a machine remains idle during the total elapsed time. The notation X_{ij} shall be used to denote the idle time of machine j between the *end* of the $(i-1)$th job and the start of the ith job.

(5) **Total Elapsed Time.** This is the time between starting the first job and completing the last job. This also includes *idle time*, if exists. It will be denoted by the symbol T. **[JNTU (B. Tech.) 2003]**

(6) **No Passing Rule.** This rule means that passing is not allowed, *i.e.* the same order of jobs is maintained over each machine. If each of the n-jobs is to be processed through two machines A and B in the order AB, then this rule means that each job will go to machine A first and then to B.

24.3. PRINCIPAL ASSUMPTIONS

(1) No machine can process more than one operation at a time.
(2) Each operation, once started, must be performed till completion.
(3) A job is an entity, *i.e.* even though the job represents a lot of individual parts, no lot may be processed by more than one machine at a time.
(4) Each operation must be completed before any other operation, which it must precede, can begin.
(5) Time intervals for processing are independent of the order in which operations are performed.
(6) There is only one of each type of machine.
(7) A job is processed as soon as possible subject to ordering requirements.
(8) All jobs are known and are ready to start processing before the period under consideration begins.
(9) The time required to transfer jobs between machines is negligible.

Q. 1. What is no passing rule in a sequencing algorithm ? Explain the principal assumptions made while dealing with sequencing problems. **[Meerut (O.R.) 90]**

2. Distinguish between "Sequencing" and "Scheduling". **[JNTU (B. Tech.) 2003]**

24.4. SOLUTION OF SEQUENCING PROBLEM

At present, solution of following cases are available :

1. n jobs and two machines A and B, all jobs processed in the order AB.
2. n jobs and three machines A, B and C, all jobs processed in the order ABC, other limitations are given in **Section 24.6.**
3. Two jobs and m machines. Each job is to be processed through the machines in a prescribed order (which is not necessarily the same for both the jobs).
4. Problems with n jobs and m-machines.

All these special cases and their solutions are explained in the following sections.

24.5. PROCESSING *n* JOBS THROUGH TWO MACHINES [Agra 99]

The problem can be described as : (*i*) only two machines A and B are involved, (*ii*) each job is processed in the order AB, and (*iii*) the exact or expected processing times $A_1, A_2, A_3, ..., A_n$; $B_1, B_2, B_3, ..., B_n$ are known (Table 24.1).

Table 24.1

Processing Times	Job (*i*)				
	1	2	3	...	n
A_i	A_1	A_2	A_3	...	A_n
B_i	B_1	B_2	B_3	...	B_n

The problem is to sequence (order) the jobs so as to minimize the total elapsed time T.

The solution procedure adopted by *Johnson* (1954) is given below.

Solution Procedure :

***Step* 1.** Select, the least processing time occurring in the list $A_1, A_2, A_3, ..., A_n$ and $B_1, B_2, B_3, ..., B_n$. If there is a tie, either of the smallest processing time should be selected.

***Step* 2.** If the least processing time is A_r, select rth job *first*. If it is B_s, do the sth job *last* (as the given order is AB).

***Step* 3.** There are now $n-1$ jobs left to be ordered. Again repeat steps I and II for the reduced set of processing times obtained by deleting processing times for both the machines corresponding to the job already assigned.

Continue till all jobs have been ordered. The resulting ordering will minimize the elapsed time T.

Proof. Since passing is not allowed, all n jobs must be processed on machine A without any *idle time* for it. On the other hand, machine B is subject to its remaining idle time at various stages. Let Y_i be the time for which machine B remains idle after completing $(i-1)$th job and before starting processing the ith job $(i = 1, 2, ..., n)$. Hence, the total elapsed time T is given by

$$T = \sum_{i=1}^{n} B_i + \sum_{i=1}^{n} Y_i$$

where some of the B_i's may be zero.

Now we wish to minimize T. However, since $\sum_{i=1}^{n} B_i$ is the total time for which machine B has to work and thus being constant, it does not form a part of minimizing T. So the problem is reduced to that of minimizing $\sum_{i=1}^{n} Y_i$. A very convenient procedure for obtaining a sequence of performing jobs so as to minimize $\sum_{i=1}^{n} Y_i$ is well explained by the following *Gantt Chart*.

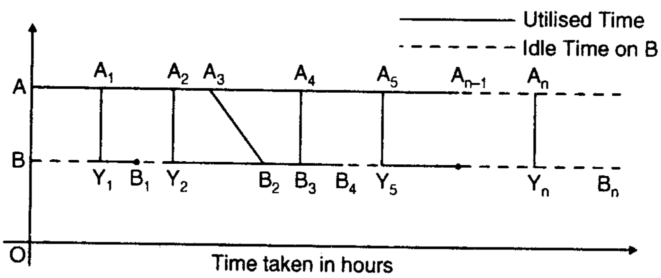


Fig. 24.1. Time taken in hours

From this chart, it is clear that

$$Y_1 = A_1$$

$$Y_2 = \begin{cases} A_1 + A_2 - Y_1 - B_1, & \text{if } A_1 + A_2 > Y_1 + B_1 \\ 0 & \text{otherwise} \end{cases}$$

The expression for Y_2 may be rewritten as

$$Y_2 = \max\{A_1 + A_2 - Y_1 - B_1, 0\}$$

Thus, $\quad Y_1 + Y_2 = \max\{A_1 + A_2 - B_1, A_1\}$, since $Y_1 = A_1$.

Similarly, $\quad Y_3 = \max\{A_1 + A_2 + A_3 - B_1 - B_2 - Y_1 - Y_2, 0\}$

$$\therefore \quad Y_1 + Y_2 + Y_3 = \max\left\{\left(\sum_{i=1}^{3} A_i - \sum_{i=1}^{2} B_i\right), \sum_{i=1}^{2} Y_i\right\}$$

$$= \max\left\{\left(\sum_{i=1}^{3} A_i - \sum_{i=1}^{2} B_i\right), \left(\sum_{i=1}^{2} A_i - B_i\right), A_1\right\}$$

In general, we get

$$\sum_{i=1}^{n} Y_i = \max\left\{\left(\sum_{i=1}^{n} A_i - \sum_{i=1}^{n-1} B_i\right), \left(\sum_{i=1}^{n-1} A_i - \sum_{i=1}^{n-2} B_i\right), ..., A_1\right\} = \max_{1 \le r \le n}\left\{\sum_{i=1}^{r} A_i - \sum_{i=1}^{r-1} B_i\right\}$$

Now, if we denote $\sum_{i=1}^{n} Y_i$ by $D_n(S)$, then the problem becomes that of finding the sequence $< S^* >$ for processing the jobs 1, 2,..., n so as to have the inequality $D_n(S^*) \le D_n(S_0)$ for any sequence $< S_0 >$ other than

$< S^* >$. In other words, we have to find the optimal sequence $< S >$ so as to minimize $D_n(S)$. This can be done iteratively by successively interchanging the consecutive jobs. Each such interchange of jobs gives a value of $D_n(S)$ less than or equal to its value before the change.

24.5-1. Johnson's Algorithm for n Jobs 2 Machines **[JNTU (Mech. & Prod.) 2004]**

The Johnson's iterative procedure for determining the optimal sequence for an n-job 2-machine sequencing problem can be outlined as follows :

Step 1. Examine the A_i's and B_i's for $i = 1, 2, ..., n$ and find out $\min_i [A_i, B_i]$

Step 2. (*i*) If this minimum be A_k for some $i = k$, do (process) the kth job first of all.
(*ii*) If this minimum be B_r for some $i = r$, do (process) the rth job last of all.

Step 3. (*i*) If there is a tie for minima $A_k = B_r$, process the kth job first of all and rth job in the last.
(*ii*) If the tie for the minimum occurs among the A_i's, select the job corresponding to the minimum of B_i's and process it first of all.
(*iii*) If the tie for minimum occurs among the B_i's, select the job corresponding to the minimum of A_i's and process it in the last. Go to next step.

Step 4. Cross-out the jobs already assigned and repeat steps 1 to 3 arranging the jobs next to first or next to last, until all the jobs have been assigned.

Q. Give Johnson's method for determining the optimal sequence for processing n jobs on two machines. **[Meerut 2002]**

24.5-2. Illustrative Example

Example 1. *There are five jobs, each of which must go through the two machines A and B in the order AB. Processing times are given (Table 24.2) below :*

Table 24.2.

	Processing time (hours)				
Job	1	2	3	4	5
Time for A	5	1	9	3	10
Time for B	2	6	7	8	4

Determine a sequence for five jobs that will minimize the elapsed time T.

Calculate the total idle time for the machines in this period. **[IAS (Main.) 95 Type; Rohilkhand 92]**

Solution. Apply steps I and II of solution procedure. It is seen that the smallest processing time is one hour for job 2 on the machine A. So list the job 2 at first place as shown below.

2				

Now, the reduced list of processing times becomes

Job	A	B
1	5	$\boxed{2}$
3	9	7
4	3	8
5	10	4

Again, the smallest processing time in the reduced list is 2 for job 1 on the machine B. So place job 1 last.

2				1

Continuing in the like manner, the next reduced list is obtained

Job	A	B
3	9	7
4	$\boxed{3}$	8
5	10	4

leading to sequence

2	4			1

and the list

Job	A	B
3	9	[7]
5	10	4

gives rise to sequence

2	4		5	1

Finally, the optimal sequence is obtained,

2	4	3	5	1

Further, it is also possible to calculate the minimum elapsed time corresponding to the optimal sequencing, using the individual processing time given in the statement of the problem. The details are given in Table 24.3.

Table 24.3

Job sequence	Machine *A*		Machine *B*	
	Time in	Time out	Time in	Time out
2	0	1	1	7
4	1	4	7	15
3	4	13	15	22
5	13	23	23	27
1	23	28	28	[30]

Thus, the minimum time, *i.e.* the time for starting of job 2 to completion of the last job 1, is 30 hrs only. During this time, the machine *A* remains idle for 2 hrs (from 28 to 30 hrs) and the machine *B* remains idle for 3 hrs only (from 0–1, 22–23, and 27–28 hrs).

The total elapsed time can also be calculated by using Gantt chart as follows :

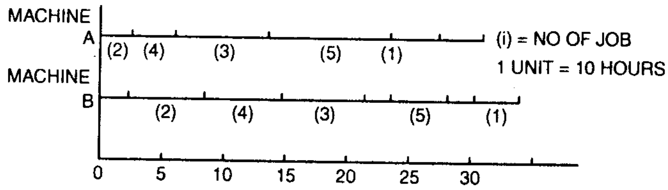


Fig. 24.2

From the *Fig*. 24.2 it can be seen that the total elapsed time is 30 hrs, and the idle time of the machine *B* is 3 hrs.

In this problem, it is observed that job may be held in inventory before going to the machine. For example, 4th job will be free on machine *A* after 4th hour and will start on machine *B* after 7th hr. Therefore, it will be kept in inventory for 3 hrs. Here it is assumed that the storage space is available and the cost of holding the inventory for each job is either same or negligible. For short duration process problems, it is negligible. Second general assumption is that the order of completion of jobs has no significance, *i.e.* no job claims the priority.

EXAMINATION PROBLEMS

1. Give three different examples of sequencing problem from your daily life. **[JNTU (B. Tech.) 2003, 02]**
2. Find the sequence that minimizes the total elapsed time required to complete the following tasks :

Tasks	:	A	B	C	D	E	F	G	H	I
Time on I machine	:	2	5	4	9	6	8	7	5	4
Time on II machine	:	6	8	7	4	3	9	3	8	11

[**Ans.** Optimum sequences are :
(*i*) A → C → I → B → H → F → D → E → G (*ii*) A → I → C → H → B → F → D → G → E
(*iii*) A → C → I → H → B → F → D → G → E (*iv*) A → I → C → B → H → F → D → E → G
(*v*) A → C → I → B → H → F → D → G → E (*vi*) A → I → C → H → B → F → D → E → G
(*vii*) A → I → C → B → H → F → D → G → E (*viii*) A → C → I → H → B → F → D → E → G
Min time = 61 hrs, idle time on I machine 11 hrs, on machine II 2 hrs.]

3. A book binder has one printing press, one binding machine, and the manuscripts of a number of different books. The time required to perform the printing and binding operations for each book are shown below. We wish to determine the order in which books should be processed, in order to minimize the total time required to turn out all the books:

Books	:	1	2	3	4	5	6
Printing time (hrs)	:	30	120	50	20	90	110
Binding time (hrs)	:	80	100	90	60	30	10

[**Ans.** Optimum Sequence : $4 \to 1 \to 3 \to 2 \to 5 \to 6$;
Min. elapsed time 430 hrs, idle time on Printing 10 hrs, on Binding 60 hrs.]

4. Following Table shows the machine time (in hours) for 5 jobs to be processed on two different machines:

Job	:	1	2	3	4	5
Machine *A*	:	3	7	4	5	7
Machine *B*	:	6	2	7	3	4

Passing is not allowed. Find the optimal sequence in which jobs should be processed. **[JNTU (B. Tech.) 2003]**
[**Ans.** $1 \to 3 \to 5 \to 4 \to 2$; min. time = 28 hrs; idle time 2 hrs for *A* and 6 hrs for *B*.]

5. Six jobs go first over machine I and then over II. The order of the completion of jobs has no significance. The following table gives the machine times in hours for six jobs and the two machines :

Job. No.	:	1	2	3	4	5	6
Machine I	:	5	9	4	7	8	6
Machine II	:	7	4	8	3	9	5

Find the sequence of the jobs that minimizes the total elapsed time to complete the jobs. Find the minimum time by using Gantt Chart or by any other method.
[**Ans.** $3 \to 1 \to 5 \to 6 \to 2 \to 4$; Total time = 42 hours.]

6. We have five jobs each of which must go through two machines in the order *AB*, Processing times are given in the table below :

Job No.	:	1	2	3	4	5
Machine *A*	:	10	2	18	6	20
Machine *B*	:	4	12	14	16	8

Determine a sequence for the five jobs that will minimize the total elapsed time.
[**Ans.** $2 \to 4 \to 3 \to 5 \to 1$; Total time 60 hours.]

7. Find the sequence that minimizes the total elapsed time required to complete the following jobs :

		Processing times in hours					
No. of jobs	:	1	2	3	4	5	6
Machine *A*	:	4	8	3	6	7	5
Machine *B*	:	6	3	7	2	8	4

[Agra 99; IAS (Maths.) 97 (Type)]

[**Ans.** $3 \to 1 \to 5 \to 6 \to 2 \to 4$; Total time = 35 hours.]

8. We have seven jobs each of which has to go through the machine M_1 and M_2 in the order M_1, M_2. Processing times (in hours) are given as :

Job	:	1	2	3	4	5	6	7
Machine M_1	:	3	12	15	6	10	11	9
Machine M_2	:	8	10	10	6	12	1	3

Determine a sequence of these jobs that will minimize the total elapsed time *T*. **[JNTU (B. Tech) 98; Agra 98]**
[**Ans.** $\begin{cases} 1 \to 4 \to 5 \to 3 \to 2 \to 7 \to 6 \\ 1 \to 4 \to 5 \to 2 \to 3 \to 7 \to 6 \end{cases}$; Total time = 67 hours.]

9. Seven jobs are to be processed on two machines *A* and *B* in the order $A \to B$. Each machine can process only one job at a time. The processing times (in hours) are as follows :

Job :	1	2	3	4	5	6	7
Machine A :	10	12	13	7	14	5	16
Machine B :	15	11	8	9	6	7	16

[Delhi (M. Com.) 97]

[**Ans.** Optimum sequence $6 \to 4 \to 1 \to 4 \to 2 \to 3 \to 5$
Min. elapsed time = 83 hrs.
Idle time : 5 hrs. for *A*, 11 hrs. for *B*]

10. A company has six jobs on hand coded '*A*' to '*F*'. All the jobs have to go thrugh two machines '*M I*' and '*M II*'. The time required for each job on each machine, in hours, is given below :

	A	B	C	D	E	F
M I	3	12	18	9	15	6
M II	9	18	24	24	3	15

Draw a sequence table scheduling the six jobs on the two machines. **[Punjabi (M.B.A.) 98]**

[**Ans.** Optimum sequence : $A \to F \to D \to B \to C \to E$

Min. elapsed time = 96 hrs. idle time : 33 hrs. for *M I* and 3 hrs. for *M II*]

11. In the machine shop, 8 different products are being manufactured each requiring time on 2 machines *A* and *B* as given below :

Product	Time (in min.) on machine A	Time (in min.) on machine B
I	30	20
II	45	30
III	15	50
IV	20	35
V	80	36
VI	120	40
VII	65	50
VIII	10	20

Decide the optimum sequence of processing of different products in order to minimize the total manufacturing time for all the products. Name and discuss the scheduling model used. **[A.I.M.A. (P.G. Dip. in Management) June 96]**

[**Ans.** Optimum sequence : VIII → III → IV → VII → VI → V → II → I

Min elapsed time = 124 hrs.]

12. There are 5 jobs, each of which must go through the two machines A and B **in the order BA**. Processing time (in hours) are given below. Determine a sequence for 5 jobs that will minimize the elapsed time.

Job	:	1	2	3	4	5
Machine A	:	3	6	8	8	4
Machine B	:	5	1	9	2	12

[JNTU (B. Tech.) 2003]

24.6. PROCESSING *n* JOBS THROUGH THREE MACHINES

The problem can be described as : (*i*) Only three machines *A*, *B* and *C* are involved, (*ii*) each job is processed in the prescribed order *ABC*, (*iii*) transfer of jobs is not permitted, *i.e.* adhere strictly the order over each machine, and (*iv*) exact or expected processing times are given in Table 24.4.

Table 24.4.

Job	Machine *A*	Machine *B*	Machine *C*
1	A_1	B_1	C_1
2	A_2	B_2	C_2
3	A_3	B_3	C_3
:	:	:	:
n	A_n	B_n	C_n

Optimal Solution. So far no general procedure is available for obtaining an optimal sequence in this case. However, the earlier method adopted by *Johnson* (1954) can be extended to cover the special cases where *either one or both* of the following conditions hold :

(*i*) The minimum time on machine $A \geq$ the maximum time on machine *B*.

(*ii*) The minimum time on machine $C \geq$ the maximum time on machine *B*.

The procedure explained here (without proof) is to replace the problem with an equivalent problem, involving *n* jobs and two fictitious machines denoted by *G* and *H*, and corresponding time G_i and H_i are defined by

$$G_i = A_i + B_i, \; H_i = B_i + C_i.$$

If this problem with prescribed ordering *GH* is solved, the resulting optimal sequence will also be optimal for the original problem.

Rules for deleting the programs which cannot be optimal.

Rule no.	Technological orderings for		Delete programs containing
	Job 1	Job 2	
I	X...Y	Y...	xy
II	X...Y...	...XY	$\bar{x}y$
III	...X...Y	...XY...	$\bar{x}y$
IV	...XY...	X...Y...	xy
V	...XY...Z...	...X...YZ...	$x\bar{y}z$
VI	...X...YZ...	...XY...Z	$\bar{x}y\bar{z}$

Here ... stands for other machines, if any. Applying the rules to this example, it is observed by taking A as X, and D as Y (I rule), that delete the programs containing ad. Such a program is 16th only. Again by II rule taking A as X and C as Y, all those programs are deleted which contain $\bar{a}c$, *i.e.*, the 5th program. Other rules are not applicable to our problem. Thus we have only following five programs.

Program No.

1	2	4	6	8
$\bar{a}$	a	a	a	a
$\bar{b}$	$\bar{b}$	b	$\bar{b}$	b
$\bar{c}$	$\bar{c}$	$\bar{c}$	c	c
$\bar{d}$	$\bar{d}$	$\bar{d}$	$\bar{d}$	d

Now finally we enumerate all these programs one by one using Gantt Chart as shown below :

From these charts it is clear that optimum program is 6th and the minimum elapsed time is 18 hours.

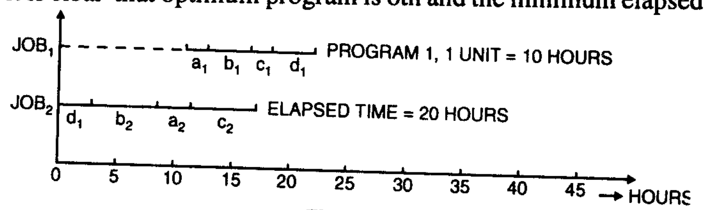


Fig. 24.3

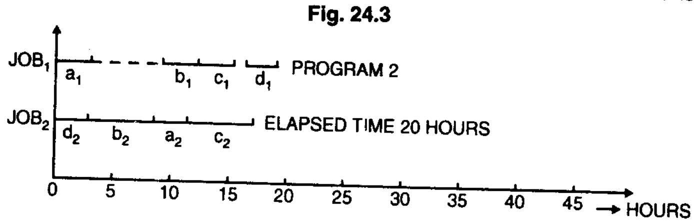


Fig. 24.4

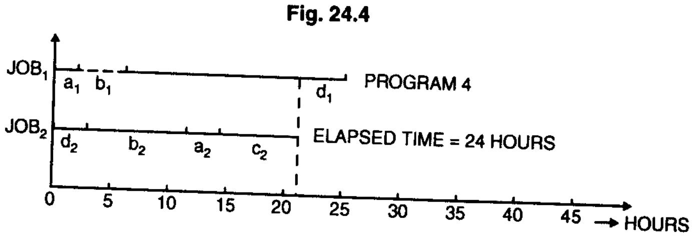


Fig. 24.5

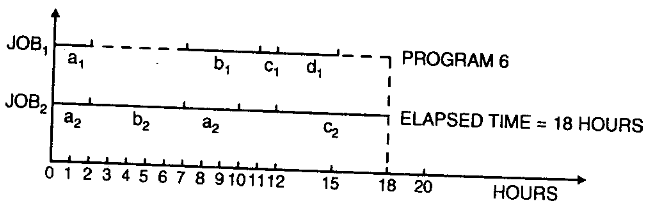


Fig. 24.6

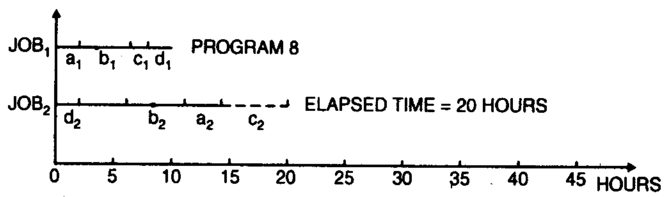


Fig. 24.7

Q. 1. Explain the problem of processing *n*-jobs through three machines. **[JNTU (BE. Tech.) 2000]**

2. Modify Johnson's method to solve the problem of processing *n* jobs through three machines. State explicitly the undertying conditions. **[Meerut 2002]**

24.6-1. Illustrative Example

Example 2. *There are five jobs, each of which must go through machines A, B and C in the order ABC. Processing times are given in Table 24.5.*

Table 24.5.

Job i	Processing Times		
	A_i	B_i	C_i
1	8	5	4
2	10	6	9
3	6	2	8
4	7	3	6
5	11	4	5

Determine a sequence for five jobs that will minimize the elapsed time T.

[JNTU (Mech. & Prod.) 2004, 2000; Banasthall (M.Sc.) 93; I.A.S. (Main) 92]

Solution. Here min $A_i = 6$, max $B_i = 6$, min $C_i = 4$.

Since one of two conditions is satisfied by min A_i = max B_i, so the procedure adopted in ***Example 1*** can be followed.

The equivalent problem, involving five jobs and two fictitious machine G and H, becomes :

Table 24.6

Job i	Processing Times	
	$G_i (= A_i + B_i)$	$H_i (= B_i + C_i)$
1	13	9
2	16	15
3	8	10
4	10	9
5	15	9

This new problem can be solved by the procedure described earlier. Because of ties, possible optimal sequences are :

(*i*)	3	2	1	4	5	(*ii*)	3	2	4	1	5
(*iii*)	3	2	4	5	1	(*iv*)	3	2	5	4	1
(*v*)	3	2	1	5	4	(*vi*)	3	2	5	1	4

It is possible to calculate the minimum elapsed time for first sequence as shown in Table 24.7.

Table 24.7.

Job	Machine A		Machine B		Machine C	
	Time in	Time out	Time in	Time out	Time in	Time out
3	0	6	6	8	8	16
2	6	16	16	22	22	31
1	16	24	24	29	31	35
4	24	31	31	34	35	41
5	31	42	42	46	46	51

Thus, any of the sequences from (*i*) to (*vi*) may be used to order the jobs through machines *A*, *B* and *C*, and they all will give a minimum elapsed time of 51 hrs. Idle time for machine *A* is 9 hrs, for *B* 31 hrs, for *C* 19 hrs.

Remarks :

1. If conditions min $A_i \geq$ max B_i and/or min $C_i \geq$ max B_i do not hold, there is no general procedure yet available.
2. The idle times for individual machines may be different for alternative optimal sequences. Although, the min. elapsed time will obviously be the same for all alternative optimal sequences.
3. If we change the ordere of machines (say, ABC → BCA), then first we must modify the given problem accordingly.

EXAMINATION PROBLEMS

1. Find the sequence that minimizes the total elapsed time required to complete the following tasks :

Tasks	A	B	C	D	E	F	G
Time on I machine	3	8	7	4	9	8	7
Time on II machine	4	3	2	5	1	4	3
Time on III machine	6	7	5	11	5	6	12

[IAS (Main) 98 (type), 96; Agra 98; Rohilkhand 94]

[Ans. (*i*) A → D → G → E → B → C → E, (*ii*) A → D → G → B → F → C → E
Min. Elapsed Time : 59 hours, Idle times : for I 13, for II 37, for III 7 hrs.]

2. We have five jobs each of which must go through the machines *A*, *B* and *C* in the order *ABC*:

Processing Times (in hours)

Job No.	1	2	3	4	5
Machine *A*	5	7	6	9	5
Machine *B*	2	1	4	5	3
Machine *C*	3	7	5	6	7

Determine a sequence for the jobs that will minimize the total elapsed time and idle time for each machine.

[Meerut (Maths) 98 BP, 93P, 82; Tamilnadue (ERODE) 97]

[Ans. (*i*) 2 → 5 → 4 → 3 → 1, (*ii*) 5 → 4 → 3 → 2 → 1, (*iii*) 5 → 2 → 4 → 3 → 1;
Min. time = 40 hours. idle times 8 hrs for A, 25 for B, and 12 hrs on C].

3. Find the sequence that minimizes the total elapsed time (in hours) required to complete the following jobs on three machines M_1, M_2 and M_3 in the order $M_1M_2M_3$.

(*a*)

		Job				
		A	B	C	D	E
Machine	M_1	4	9	8	6	5
	M_2	5	6	2	3	4
	M_3	8	10	6	7	11

[Meerut 2002]

[Ans. (*i*) A → D → E → B → C (*ii*) A → E → D → B → C (*iii*) D → A → E → B → C
(*iv*) D → E → A → B → C (*v*) E → D → A → B → C (*vi*) E → A → D → B → C
Min. time = 51 hours; Idle times : for I 9 hrs, for II 31 hrs, for III 19 hrs.]

(*b*)

		Job				
		A	B	C	D	E
Machine	M_1	5	7	6	9	5
	M_2	2	1	4	5	3
	M_3	3	7	5	6	7

[Ans. B → E → D → C → A, or E → B → D → C → A; Min. time = 40 hours.]

4. Find the sequence that minimizes the total elapsed time required to complete the following tasks. Each job is processed in the order *ACB*.

		Job						
		1	2	3	4	5	6	7
Machines	A	12	6	5	11	5	7	6
	B	7	8	9	4	7	8	3
	C	3	4	1	5	2	3	4

[Ans. (*i*) 3 → 5 → 2 → 6 → 1 → 4 → 7, (*ii*) 3 → 5 → 6 → 2 → 1 → 4 → 7;
Min. time = 59 hrs, Idle time for *A* = 7 hrs, for *B* = 113 hrs, for *C* = 37 hrs.] **[JNTU (Mech. & Prod.) 2004]**

5. A company has six jobs which go through three machines *X*, *Y* and *Z* in the order *XYZ*. The processing time in minutes for each job on each machine is as follows :

		Job					
		1	2	3	4	5	6
Machine	X	18	12	29	36	43	37
	Y	7	12	11	2	6	12
	Z	19	12	23	47	28	36

What should be the sequence of the jobs. ? **[JNTU (B. Tech.) Comp. Sc. 2003, (Mech.) 99; Meerut (MA) 97 P]**
[**Ans.** 2 → 1 → 4 → 6 → 5 → 3; Min. time = 209 min; Idle time 34 min. for *X*, 159 min. for *Y* and 44 min. for *Z*].

6. With the help of the following example describe the algorithm for solving an '*n*-job, three-machine' sequencing problem :

Job	Processing time (in hours)		
	Machine *A*	Machine *B*	Machine *C*
1	3	3	5
2	8	4	8
3	7	2	10
4	5	1	7
5	2	5	6

[Delhi (M. Com.) 96]

24.7. PROCESSING TWO JOBS THROUGH *m* MACHINES

Consider the situation when, (*i*) there are *m* machines, denoted by $A_1, A_2, A_3, ..., A_m$, (*ii*) one has to perform only two jobs *i.e.*, job 1 and job 2, (*iii*) the technological ordering of each of the two jobs through *m* machines is known in advance. Such ordering may not be the same for both the jobs, (*iv*) the exact or expected processing times $A_1^{(1)}, A_2^{(1)}, A_3^{(1)}, \ldots, A_m^{(1)}, A_1^{(2)}, A_2^{(2)}, A_3^{(2)}, \ldots, A_m^{(2)}$ are known.

The problem is to minimize the total elapsed time *T* (the time from the start of the first job to the completion of the second job.).

24.7-1. Graphical Method

In the two-job *m*-machine problem, there is a graphical procedure which is rather simple to apply and usually provides good (though not necessarily optimal) results. The following example will make the graphical procedure clear.

Example 3. *Use graphical method to minimize the time needed to process the following jobs on the machines shown below, i.e. for each machine find the job which should be done first. Also calculate the total time needed to complete both the jobs.* **[Virbhadra 2000; Meerut (Maths.) 96]**

Table 24.8.

Job 1	Sequence of Machines	:	A	B	C	D	E
	Time	:	2	3	4	6	2
Job 2	Sequence of Machines	:	C	A	D	E	B
	Time	:	4	5	3	2	6

Solution.

Step 1. First, draw a set of axes, where the horizontal axis represents processing time on job 1 and the vertical axis represents processing time on job 2 (Fig. 24.8).

Step 2. Layout the machine time for two jobs on the corresponding axes in the given technological order (Fig. 24.8)

Machine A takes 2 hrs for job 1 and 5 hrs for job 2. Construct the rectangle *PQRS* for the machine *A*. Similarly, other rectangles for machines *B, C, D* and *E* are constructed as shown in Fig. 24.8.

Step 3. Make programme by starting from the origin *O* and moving through various states of completion (points) until the point marked

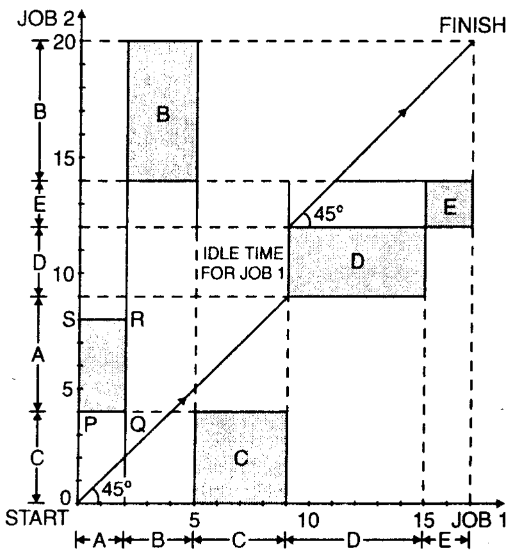


Fig. 24.7 Graphical solution for the 2-job 5-machine sequencing problem.

'finish' is obtained. Physical interpretation of the path thus chosen involves the series of segments which are horizontal or vertical or diagonal making an angle of 45° with the horizontal. Moving to the right means that job 1 is proceeding while job 2 is idle, and moving upward means that job 2 is proceeding while job 1 is idle, and moving diagonally means that both the jobs are proceeding simultaneously.

Further, both the jobs cannot be processed simultaneously on the same machine. Graphically, diagonal movement through the blocked-out (shaded) area is not allowed, and similarly for other machines too.

Step 4. **To find an optimal path.** An optimal path (programme) is one that minimizes idle time for job 1 (horizontal movement). Similarly, an optimal path is one that minimizes idle time for job 2 (vertical movement). Choose such a path on which diagonal movement is as much as possible. According to this, choose a good path by inspection as shown in Fig. 24.7 by arrows.

Step 5. **To find the elapsed time.** The elapsed time is obtained by adding the idle time for either of the job to the processing time for that job.

In this problem, the idle time for the chosen path is seen to be 3 hrs. for the job 1, and zero for the job 2. Thus, the total elapsed time, $17 + 3 = 20$ hrs is obtained. **[Meerut (OR) 2003]**

EXAMINATION PROBLEMS

1. Use graphical method to minimize the time needed to process the following jobs on the machines shown, *i.e.* for each machine find the job which should be done first. Also, calculate the total time needed to complete both the jobs.

			Machines				
Job 1	Sequence	:	A	B	C	D	E
	Time	:	3	4	2	6	2
Job 2	Sequence	:	B	C	A	D	E
	Time	:	5	4	3	2	6

[Meerut (OR) 2003; JNTU (Mech.) 98, 97; Agra 99, 93]

[**Ans.** (*i*) A(1), B(2), C(2) D(2), E(1); (*ii*) Total time = 22 hrs, idle time for job 1 = (2 + 3) hrs, for job 2 = 2 hrs.]

2. Two jobs are to be processed on four machines A, B, C and D. The technological order for these machines is as follows :

Job 1	:	A	B	C	D
Job 2	:	D	B	A	C

Processing times are given in the following table :

	Machines			
	A	B	C	D
Job 1	4	6	7	3
Job 2	4	7	5	8

Find the optimal sequence of jobs on each of the machine.

[**Hint.** An optimal path is one that minimizes the idle time for job 1 (horizontal movement). Similarly, an optimal path is one that minimizes idle time for job 2 (vertical movement).]

3. A Machine shop has four machines A, B, C, D. Two jobs must be processed through each of these machines. The time (in hours) taken on each of the machines and the necessary sequence of jobs through the shop are given below :

Job 1	Sequence	:	A	B	C	D
	Time	:	2	4	5	1
Job 2	Sequence	:	D	B	A	C
	Time	:	6	4	2	3

Use graphic method to obtain minimum elapsed time.

[**Ans.** Graph will show that both the jobs can be done simultaneously until job 1 is completed, and then job 2 is completed. The total elapsed time is 15 hours.]

4. Use graphical method to minimize the time needed to process the following jobs on machines A, B, C, D and E. Find the total time elapsed to complete both the jobs. Also find for each job the machines on which it should be processed. Processing time given is in minutes.

Job 1 :	A–20	C–10	D–10	B–30	E–25	F–15
Job 2 :	A–10	C–30	B–15	D–15	F–10	E–20

[AIMS (BE) Bangl. 2002]

24.8. PROCESSING n JOBS THROUGH m MACHINES

Let each of the n jobs be processed through m machines, say $M_1, M_2, M_3, ..., M_m$ in the order $M_1M_2M_3 ... M_m$, and T_{ij} denote the time taken by the ith machine to complete the jth job. The step-by-step procedure for obtaining an optimal sequence is as follows :

Step 1. First find, (*i*) $\min_j (T_{1j})$, (*ii*) $\min_j (T_{mj})$, and (*iii*) $\max_j (T_{2j}, T_{3j}, ..., T_{(m-1)j})$ for $j = 1, 2, ..., n$.

Step 2. Then check whether
(*i*) $\min_j (T_{1j}) \geq \max_j (T_{ij})$ for $i = 2, 3, ..., m-1$, or(*ii*) $\min_j (T_{mj}) \geq \max_j (T_{ij})$ for $i = 2, 3, ..., m-1$.

Step 3. If inequalities of *step* 2 are not satisfied, this method fails. Hence go to next step.

Step 4. Convert the m-machine problem into 2-machine problem considering two fictitious machines G and H, so that

$$T_{Gj} = T_{1j} + T_{2j} + ... + T_{(m-1)j} \text{ and } T_{Hj} = T_{2j} + T_{3}j + ... + T_{mj}.$$

Now determine the optimal sequence of n jobs through 2 machines by using the optimal sequence algorithm.

Step 5. In addition to conditions given in *step* 4, if $T_{2j} + T_{3j} + ... + T_{(m-1)j} = C$ (*a fixed positive constant*) for all $j = 1, 2, ..., n$, then determine the optimal sequence for n jobs and two machines M_1 and M_m in the order M_1M_m by using the optimal sequence algorithm.

Notes :

1. If in addition to the condition given in the *step* 4, $T_{1j} = T_{mj}$ and $T_{Gj} = T_{Hj}$, for $j = 1, 2, ..., n$. then n! number of optimal sequences will exist.
2. This procedure for sequencing n jobs through m machines is not a general procedure. This method is applicable to only such sequencing problems in which minimum time or cost of processing the jobs through first and/or last machine is greater than or equal to the time or cost of processing the jobs through mediocre machine.

Example 4. *Solve the following sequencing problem giving an optimal solution when passing is not allowed.* **[Agra 99, 98; Meerut 93]**

		Job (j)				
		A	B	C	D	E
	M_1	11	13	9	16	16
Machine	M_2	4	3	5	2	6
(i)	M_3	6	7	5	8	4
	M_4	15	8	13	9	11

Solution. In this example,

$$\min_j T_{1j} = 9 = T_{13}, \min_j T_{4j} = 8 = T_{42} ; \max_j T_{2j} = 6 = T_{25}, \max_j T_{3j} = 8 = T_{34}.$$

Since the conditions

$$(\min_j T_{1j} \geq \max_i T_{ij}, i = 2, 3) \text{ and } (\min_j T_{4j} \geq \max_j T_{ij}, i = 2, 3)$$

are satisfied, convert this problem into five jobs and two-machine problem.

Also, $M_{2j} + M_{3j} = 10$, a fixed positive constant for all $j = 1, 2, ..., 5$, therefore the problem will reduce to an optimal sequence for five jobs and two machines M_1 and M_4 in the order M_1M_4 meaning thereby M_2 and M_3 have no effect on the optimality of the sequences. Following the usual optimal sequence algorithm, obtain the optimal sequence C → A → E → D → B. Therefore, the total elapsed time may be calculated as follows :

		Machine			
		M_1	M_2	M_3	M_4
	C	0–9	9–14	14–19	19–32
Job	A	9–20	20–24	24–30	32–45
	E	20–36	36–42	42–46	46–57
	D	36–52	52–54	54–62	62–71
	B	52–65	65–68	68–75	75–83

This table shows that the total elapsed time is 83 hrs.

Example 5. *There are 4 jobs each of which has to go through the machines M_i, $i = 1, 2, ..., 6$ in the order $M_1 M_2 ... M_6$. Processing times are given.*

		Machine (i)					
		M_1	M_2	M_3	M_4	M_5	M_6
Job (j)	A	20	10	9	4	12	27
	B	19	8	11	8	10	21
	C	13	7	10	7	9	17
	D	22	6	5	6	10	14

Determine a sequence of these four jobs which minimizes the total elapsed time T.

[Agra 98; Rohilkhand 91 (Type)]

Solution. In this example,

$$(\min_j T_{1j} = 13, \min_j T_{6j} = 14) \text{ and } (\max_j T_{2j} = 10, \max_j T_{3j} = 11, \max_j T_{4j} = 8, \max_j T_{5j} = 12).$$

Since the conditions :

$$(\min_j T_{1j} \geq \max_j T_{ij}) \text{ and } (\min_j T_{6j} \geq \max_j T_{ij}) \text{ for } i = 2, 3, 4, 5$$

are satisfied, the problem can be converted into 4-job and 2-machine problem.

Thus if G and H are two fictitious machines such that

$$T_{Gj} = \sum_{i=1}^{m-1} T_{ij} \text{ and } T_{Hj} = \sum_{i=2}^{m} T_{ij},$$

then the problem can be reformulated as 4-job and 2-machine problem :

		Jobs			
		A	B	C	D
Machines	G	55	56	46	49
	H	62	58	50	41

Using the optimal sequence algorithm, an optimal sequence is obtained as C → A → B → D. The total elapsed time may be calculated as :

		Machines					
		M_1	M_2	M_3	M_4	M_5	M_6
Jobs	C	0–13	13–20	20–30	30–37	37–46	46–63
	A	13–33	33–43	43–52	52–56	56–68	68–95
	B	33–52	52–60	60–71	71–79	79–89	95–116
	D	52–74	74–80	80–85	91–101	91–101	116–130

This table shows that the total elapsed time is 130 hrs.

EXAMINATION PROBLEMS

When passing is not allowed, solve the following problem giving an optimal solution.

1.

		Machine				
		M_1	M_2	M_3	M_4	M_5
Job	A	9	7	4	5	11
	B	8	8	6	7	12
	C	7	6	7	8	10
	D	10	5	5	4	8

[Ans. A → C → B → D; min. time = 67 hours] **[Meerut 2002 (Type)]**

2.

		Machine			
		M_1	M_2	M_3	M_4
Job	A	24	7	7	29
	B	16	9	5	15
	C	22	8	6	14
	D	21	6	8	32

[Ans. D → A → B → C; min. time = 125 hrs.]

3.

		Machine			
		M_1	M_2	M_3	M_4
	A	15	5	4	15
Job	B	12	2	10	12
	C	16	3	5	16
	D	17	3	4	17

[Meerut (M.Sc. Maths) 96, 90]

[**Ans.** 4 ! (= 24 sequences; min. time = 84 hrs.)]

EXAMINATIONS REVIEW QUESTIONS

1. Give Johnson's procedure for determining an optimal sequence for processing n items on two machines. Give justification of the rule used in the procedure.
2. Write short notes on the following :
   (*i*) Sequencing decision problem for *n* jobs on two machines.
3. What is Gantt chart ? Illustrate with an example.
4. Give a mathematical formulation of the optimal assignment and travelling-salesman problems. In what way do the feasible solution matrices in the two cases differ ?
5. Explain briefly the solution procedure of processing of two jobs through four machines when the technological ordering of each of the jobs through the machines is prescribed in advance. Establish the following rule : "If machine *A* precedes machine *B* for Job 1 and machine *B* precedes machine *A* for Job 2, then no programme that contains both the decisions : (*a*) job 2 before job 1, and (*b*) job 1 before job 2 on machine *B*, is technologically feasible."
6. (*a*) A job consists of *N* steps. Step *i* takes time t_i. If these jobs are grouped somehow into station systems, then twice as many units can be produced each day. Also two set-ups in parallel can also double the production rate, critically examine the advantages of these two approaches.
   (*b*) What do you understand by the following terms in the context of sequence of jobs :
   (*i*) Job arrival pattern. (*ii*) Number of machines. (*iii*) The flow pattern in the shop,
   (*iv*) The criteria for evaluating the performance of a schedule.
   (*c*) By using appropriate notation, obtain formulae for the following :
   (*i*) Waiting time of a job, (*ii*) Completion time of a job
   (*iii*) Mean flow time, (*iv*) Mean lateness.

EXAMINATIONS REVIEW PROBLEMS

1. A ready-made garment manufacturer has to process seven items through two stages of production, *viz.* cutting and sewing. The time taken for each of these items at the different stages are given below in appropriate units.

Item No.	:	1	2	3	4	5	6	7
Processing Time Cutting	:	5	7	3	4	6	7	12
Processing Time Sewing	:	2	6	7	5	9	5	8

(*a*) Find an order in which these items are to be processed through these stages so as to minimize the total processing time.

[**Ans.** 3 → 4 → 5 → 7 → 2 → 6 → 1; min. time = 46 hrs, idle time on cutting = 2 hrs, idle time on sewing = 4 hrs.]

(*b*) Suppose the third stage of production is added, *viz.* pressing and packing with processing time for these items as follows :

Item	:	1	2	3	4	5	6	7
Processing time (Pressing & Packing)	:	10	12	11	13	12	10	11

minimize the time taken to process all the items through all the three stages. [I.A.S. (Main) 91]

[**Hint.** When third stage is added, this becomes 3-machine 2-job problem. Required condition is also satisfied.]

[**Ans.** 1 → 4 → 3 → 6 → 2 → 5 → 7; min. time = 86 hrs. idle time on cutting 42 hrs. on sewing 44 hrs, and 7 hrs on pressing and packing.]

2. The following table gives the processing time of 10 items on 2machines A and B. Each item has to be processed first on machine A and then on machine B. Use Johnson's Algorithm to prepare a processing schedule which takes the least processing time. Find this time also.

Item	:	1	2	3	4	5	6	7	8	9	10
Machine A	:	3	2	13	10	5	6	2	15	10	7
Machine B	:	5	8	5	12	11	10	13	7	5	12

[JNTU (B. Tech) 2002 (Type)]

3. Find the sequence, for the following eight jobs, that will minimize the total elapsed time for the completion of all the jobs. Each job is processed in the same order CAB.

Jobs

		1	2	3	4	5	6	7	8
Machines	A	4	6	3	4	5	3	6	2
	B	8	10	7	8	11	8	9	13
	C	5	6	2	3	4	9	15	11

The entries give the time in hours on the machines.
[Ans. 4 → 1 → 3 → 5 → 2 → 7 → 8 → 6; min. time = 81 hrs.]

4. Find the sequence that minimizes the total time required in performing the following jobs on three machines in the order ABC.

Job

	1	2	3	4	5	6
A	8	3	7	2	5	1
B	3	4	5	2	1	6
C	8	7	6	9	10	9

5. A company has jobs which must go through machines *X*, *Y* and *Z* in the order *XYZ*. The processing times are :

Job

		1	2	3	4
Machine	X	30	120	50	20
	Y	80	100	90	50
	Z	40	60	50	120

What should be the sequence of jobs ?

6. Find the optimal sequence for processing 4 jobs, *A, B, C, D* on four machines A_1, A_2, A_3, A_4 in the order A_1 A_2 A_3 A_4. Processing times for which are given as under :

Processing times (a_{ij}) in hours

Job/machine	A_1 (a_{i_1})	A_2 (a_{i_2})	A_3 (a_{i_3})	A_4 (a_{i_4})
A	15	5	4	14
B	12	2	10	12
C	13	3	6	15
D	16	0	3	19

[Meerut (Maths) 98 BP]

7. (*a*) A book binder has one printing press, one binding machine and the manuscripts of a number of different books. The times required to perform the printing and binding operation for each book are known. Determine the order in which the books should be processed in order to minimize the total time required to process all the books. Find also the total time required (clearly state any agorithm you might use)

Processing time in minutes

Book	:	1	2	3	4	5
Printing time	:	40	90	80	60	50
Binding time	:	50	60	20	30	40

(*b*) Suppose that an additional operation is added to the process described in (*a*) : finishing. The time required for this operation are given below.

Finishing time in minutes

Book	:	1	2	3	4	5
Finishing	:	80	100	60	70	110

What is the order in which the books should be processed. Find also the minimal total elapsed time.

[Ans.(*a*) 1 → 2 → 5 → 4 → 3; min. time = 340 minutes. Idle time on printing machine = 20 min., on binding machine = 110 minutes.

(*b*) 4 → 1 → 5 → 2 → 3; min. time = 510 minutes. Idle time on printing machine = 190 min, on binding machine = 310 minutes, and 90 minutes on finishing machine].

8. A machine operator has to perform three operations : turning, threading and knurling on different jobs. The time required to perform these operations (in minutes) for each job is known. Determine the order in which the jobs should be processed in order to minimized the total time required to work out all the jobs. Also find the minimum elapsed time.

	1	2	3	4	5	6
I	3	12	5	2	9	11
II	6	6	4	6	3	1
III	13	14	9	12	8	13

[JNTU (B. Tech.) 2003]

9. Suppose several types of furniture must pass through three pre-finishing stage in the same order. Because of the size and complexity of each type of furniture the processing time at each stage varies considerably.

Stage / Type	Stage 1	Stage 2	Stage 3
Chair	7	2	5
Desk	10	3	8
Lamp	6	4	4
Table	7	5	2
Coffee table	8	5	2

(i) Determine an optimal sequence.

(ii) What is the total elapsed time for an optimal sequence.

(iii) What is the total idle time at stages 1, 2 and 3. **[JNTU (B. Tech.) 2003]**

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25

PROJECT MANAGEMENT BY PERT-CPM

25.1. INTRODUCTION

A project defines a combination of interrelated activities which must be executed in a certain order before the entire task can be completed. The activities are interrelated in a logical sequence in such a way that some activities cannot start until some others are completed. An activity in a project is usually viewed as job requiring time and resources for its completion. Until recently, planning was seldom used in the design phase. As the technological development took place at a very rapid speed and the designs become more complex with more inter-departmental dependence and interaction, the need for planning in the development phase become inevitable.

Until five decades ago, the best known 'planning tool' was the so called *Gantt bar chart* which specifies the start and finish times for each activity on a horizontal time-scale, but the disadvantage is the interdependency between different activities (that mainly controls the progress of the project) which cannot be determined from the bar chart. Growing complexities of modern projects have demanded more systematic and effective planning techniques with the objective of optimizing the efficiency of executing the project. Efficiency implies effecting the utmost reduction in the time required to complete the project while accounting for economic feasibility of using available resources. Project management has evolved as a new field with the development of *two* 'analytic' techniques for planning, scheduling and controlling of projects. These are the *Critical Path Method (CPM)* and the *Project Evaluation and Review Technique (PERT).*

25.2. HISTORICAL DEVELOPMENT OF CPM / PERT TECHNIQUES

In 1956-58, above two techniques were developed by two different groups almost simultaneously. CPM was developed by Walker from E.L. du pont de Nemours Company to solve project scheduling problems and was later extended to a more advanced status by Mauchly Associates. During the same time, PERT was developed by the team of engineers working on the polar's Missile programme of US Navy. This was a large project involving many departments and there were many activities about which they had a very little information about the duration of the project. Under such conditions, the project was to be completed within a specified time. To coordinate activities of various departments, this group used PERT and devised the technique independent of CPM.

The methods are essentially network-oriented techniques using the same principle. PERT and CPM are basically time-oriented methods in the sense that they both lead to the determination of a time schedule for the project. The significant difference between two approaches is that the time estimates for the different activities in CPM were assumed to be deterministic while in PERT these were described probabilistically. Now a days, PERT and CPM actually comprise one technique and the differences, if any are only historical. Therefore, these techniques are referred to as '*project scheduling*' techniques.

Q. Distinguish between PERT and CPM techniques. **[VTU (BE Mech.) 2002]**

25.3. APPLICATIONS OF PERT / CPM TECHNIQUES

These methods have been applied to a wide variety of problems in industries and have found acceptance even in government organizations.

These include :

(*i*) construction of a dam or canal system in a region, (*ii*) construction of a building or highway,
(*iii*) maintenance or overhaul of aeroplanes or oil refinery, (*iv*) space flight,
(*v*) cost control of a project using PERT/COST, (*vi*) designing a prototype of a machine,
(*vii*) development of supersonic planes.

25.4. BASIC STEPS IN PERT/CPM TECHNIQUES

Project scheduling by PERT/CPM consists of *four* main steps :

1. Planning. The planning phase is started by splitting the total project into small projects. These smaller projects, in turn, are divided into activities and are analysed by the department or a section. The relationship of each activity with respect to other activities are defined and established, and the corresponding responsibilities and the authority are also stated. Thus, the possibility of overlooking any task necessary for the completion of the project is reduced substantially.

2. Scheduling. The ultimate objective of the scheduling phase is to prepare a time chart showing the start and finish times for each activity as well as its relationship to other activities of the project. Moreover, the schedule must pinpoint the critical path (in view of time) activities which require special attention if the project is to be completed in time. For non-critical activities, the schedule must show the amount of slack or float times (defined later) which can be used advantageously when such activities are delayed or when limited resources are to be utilized effectively. In this phase, it is possible to resource requirements such as time, manpower, money machines, etc.

3. Allocation of Resources. Allocation of resources is performed to achieve the desired objective. A resource is a physical variable such as labour, finance,equipment and space which will impose a limitation on time for the project. When resources are limited and conflicting, demands are made for the same type of resources a systematic method for allocation of resources become essential. Resource allocation usually incurs a compromise, and the choice of this compromise depends on the judgement of managers.

4. Controlling. The final phase in project management is *controlling.* Critical path methods facilitate the application of the priciple of management by expectation to identifying areas that are critical to the completion of the project. By having progress reports from time to time and updating the network continuously, a better financial as well as technical control over the project is exercised. Arrow diagrams and time charts are used for making periodic progress reports. If necessary, new course of action is determined for the remaining portion of project.

25.5 NETWORK DIAGRAM REPRESENTATION

In project scheduling, the first step is to sketch an arrow diagram which shows inter-dependencies and the precedence relationship among activities (as defined below) of the project. In a network representation of a project, certain basic definitions are used.

1. Activity. Any individual operation, which utilises resources and has an end and a beginning, is called *activity.* An arrow is commonly used to represent an activity with its head indicating the direction of progress in the project. These are usually classified into following *four* categories :

(*i*) ***Predecessor activity.*** Activities that must be completed immediately prior to the start of another activity are called *predecessor activities.*

(*ii*) ***Successor activity.*** Activities that cannot be started until one of more of other activities are completed, but immediately succeed them are called *successor activities.*

(*iii*) ***Concurrent activity.*** Activities which can be accomplished concurrently are known as concurrent activities. It may be noted that an activity can be a predecessor or a successor to an event or it may be concurrent with one or more of the other activities.

(*iv*) ***Dummy activity.*** An activity which does not consume any kind of resource but merely depicts the technological dependence is called a *dummy activity.*

It may be noted that the dummy activity is inserted in the network to clarify the activity pattern in the following *two* situations :

(*i*) to make activities with common starting and finishing points distinguishable, and

(*ii*) to identify and maintain the proper precedence relationship between activities that are not connected by events.

For example, consider a situation where A and B are concurrent activities, C is dependent on A, and D is dependent on A and B both. Such a situation can be handled by using a dummy activity as follows (*Fig.* 25.1.) :

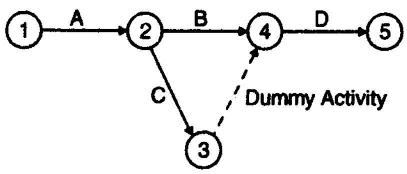


Fig. 25.1.

In another situation, consider the following diagram where job B and C have the same job reference and they can be started independently on completion of A. But, D could be started only after completion of B and C. This relationship is shown by the dotted line (*Fig.* 25.2.).

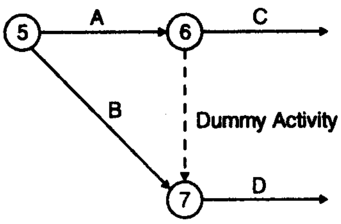


Fig. 25.2.

2. Event. An event represents a point in time signifying the completion of some activities and the beginning of new ones. This is usually represented by a circle 'O' in a network which is also called a *node* or *connector.*

The events can be further classified into following three categories (as shown below in the figure) :

(*i*) **Merge event.** When more than one activity comes and joins an event, such event is known as *merge event.*

(*ii*) ***Burst event.*** When more than one activity leaves an event, such event is known as a *burst event.*

(*iii*) **Merge and burst event.** An activity may be a merge and burst event at the same time as with respect to some activities it can be a merge event and with respect to some other activities it may be a burst event.

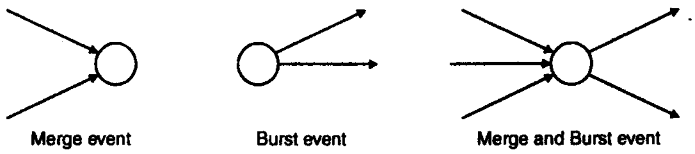


Fig. 25.3.

Remarks :

1. An event is that particular instant of time at which some *specific part* of a project has been or is to be achieved. While, an activity is actual performance of a task. An activity requires time and resources for its completion.

   Examples of events : design completed, pipe line laid,etc.

   Examples of activities : assembly of parts, mixing of concrete, preparing budget, etc.

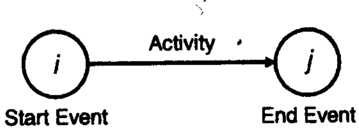


Fig. 25.4.

2. Events are described by such words as : complete, start, issue, approve, tested. etc.

   While, the word like : design, procure, test, develop, prepare etc. shows that work is being accomplished and thus represent activities.
3. While drawing networks, it is assumed that (*i*) time flows from left to right, and (*ii*) head events always have number higher than that of tail event. Thus activity (*i* – *j*) always means that the job which begins at event '*i*' is completed at event '*j*'
4. Network representation is based on the following *two* axioms :

   (*i*) An event is not said to be complete until all the activities flowing into it are completed.

   (*ii*) No subsequent activity can begin until its tail event is reached or completed.

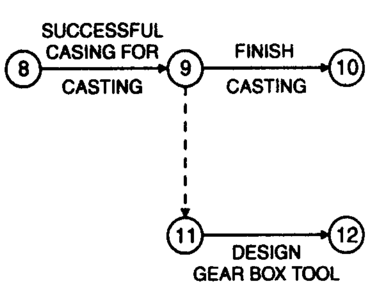


Fig. 25.5

Illustration. Designing tools for a gear box is an *activity.* A decision to start designing tools may depend on having a successful casing for gear-box casting.

In terms of technological sequence, casting as such has little bearing on the tooling of the gear box, but the management would prefer to have a successful casing before gear-box tool is designed. Thus the dependence of gear-box tools on successful casing is shown as a dummy activity (*Fig.* 25.5.).

It is also important to note that the length of the arc (or arrow) need not be proportional to the duration of the activity nor does it have to be drawn as a straightline.

If the duration of each activity as well as their logical sequence is known, it can be shown on a network (Fig. 25.6). The duration may be measured in days.

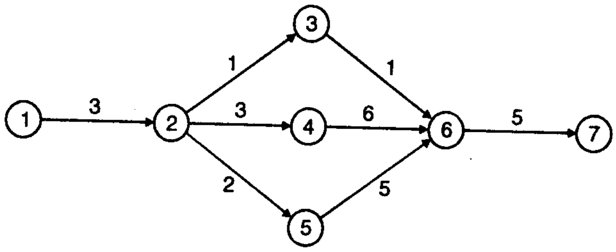


Fig. 25.6. Activity duration on the network

3. Sequencing. The first prerequisite in the development of a network is to maintain the precedence relationships. In order to make a network, following points should be taken into consideration :

(*i*) What job or jobs precede it ? (*ii*) What job or jobs could run concurrently ?

(*iii*) What job or jobs follow it ? (*iv*) What controls the start and finish of a job ?

Since all further calculations are based on the network, it is necessary that a network be drawn with full care. There are many ways to draw a network. In this text, the method will be used which follows the precedence table. It is generally agreed that dummy activities be used as liberally as needed in the first attempt, while revising the same network, every attempt should be made to minimize them.

The following example of water pump shows basic steps required in drawing a network.

Illustration. *A new type of water pump is to be designed for an automobile. Major specifications are given. Following list represents major activities for effective control of the project :*

(*i*) Drawings prepared and approved (*ii*) Cost analysis (*iii*) Tool feasibility (economics)

(*iv*) Tool manufactured (*v*) Favourable cost (*vi*) Raw materials procured

(*vii*) Sub-assemblies ordered (*viii*) Sub-assemblies received (*ix*) Parts manufactured

(*x*) Final assembly (*xi*) Testing and shipment.

Based on this available information, a precedence table may be formed (Table 25.1).

Precedence Table 25.1.

Activity	Description	Preceding Activity
A	Drawing prepared and approved	–
B	Cost analysis	A
C	Tool feasibility (economics)	A
D	Tool manufactured	C
E	Favourable cost	B, C
F	Raw materials procured	D, E
G	Sub-assemblies ordered	E
H	Sub-assemblies received	G
I	Parts manufactured	D, F
J	Final assembly	I, H
K	Testing and shipment	J

In this table, due consideration has been given to precedings of an activity/activities. While drawing the network, other factors will be considered.

Step 1. The activity 'A' has no preceding activity and is represented by an arrowed line (*Fig* 25.7).

Step 2. Activities B and C are preceded by an activity 'A' and activities B and C could be done concurrently (if resources are not binding). No other activity can be scheduled at this stage. This is shown in *Fig.* 25.8.

Step 3. The activity 'D' can be sequenced easily. Cost favourable activity 'E' cannot be scheduled unless activities B and C are scheduled. Further, it is observed that dependence of 'cost favourable activity' on the *'economy of tooling'* is from a *'technical view-point'* and does not consume any resource, and

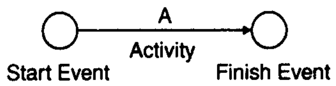


Fig. 25.7

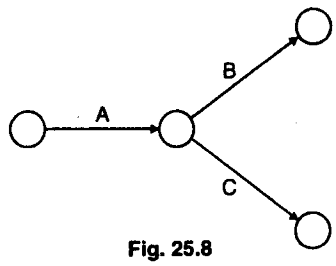


Fig. 25.8

hence this *dummy* activity should be shown by a dotted line. Thus, the network up to this stage is shown in Fig. 25.9.

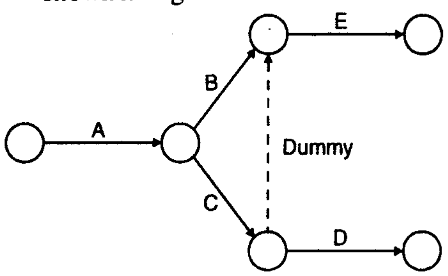


Fig. 25.9

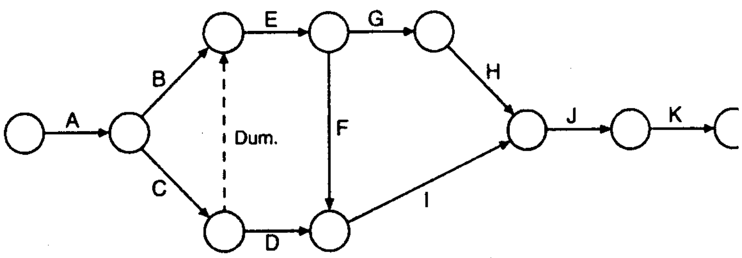


Fig. 25.10

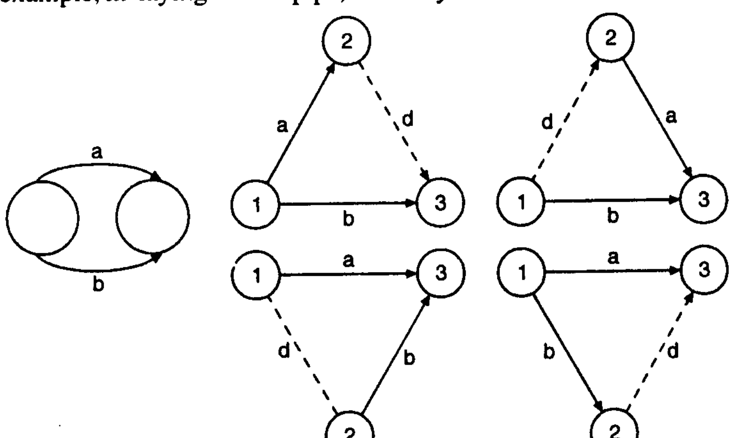
Rest of the network is straight forward, and thus the complete network is shown is Fig 25.10.

25.6. RULES FOR DRAWING NETWORK DIAGRAM

Rules for drawing network diagrams are summarized as follows :

Rule 1. ***Each activity is represented by one and only one arrow in the network.***

This implies that no single activity can be represented twice in the network. This is to be distinguished from the case where one activity is broken into segments. In such a case, each segment may be represented by a separate arrow. For example, in layingdown a pipe, this may be done in sections rather than as one job.



(a) (b)

Fig. 25.11

Rule 2. ***No two activities can be indentified by the same end events.***

For example, activities *a* and *b* (*Fig* 25.11*a*) have the same end events.The procedure is to introduce a *dummy activity* either between *a* and one of end events or between *b* and one of the end events. Modified representations after introducing the dummy *d* are shown in *Fig.* 25.11*b*. As a result of using the dummy *d*, activities *a* and *b* can now be identified by unique end events. It must be noted that a dummy activity does not consume any time or resources.

Dummy activities are also useful in establishing logic relationship in the arrow diagram which otherwise cannot be represented correctly. Suppose jobs *a* and *b* in a certain project must precede the job *c*. On the other hand, the job *e* is preceded by the job *b* only. *Fig*.25.12 (a) shows the incorrect way since, though the relationship between *a*, *b* and *c* are correct, the diagram implies that the job must be preceded by both the jobs *a* and *b*. The correct representation using the dummy *d* is shown obvious that indicated precedence relationships are justified.

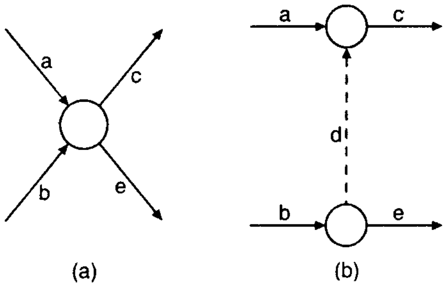


Fig. 25.12

Rule 3. ***In order to ensure the correct precedence relationship in the arrow diagram, following questions must be checked whenever any activity is added to the network.***

(*i*) ***What activity must be completed immediately before this activity can start ?***

(*ii*) ***What activities must follow this activity ?***

(*iii*) ***What activities must occur simultaneously with this activity ?***

These rules, have already been illustrated by an illustration on page 943. Apart from this, a few important suggestions for drawing good networks are given below.

Suggestions. If two or more individuals draw the same network for a given project, it seldom happens that some of them will look alike. As a matter of fact, no two of them may even look similar. The reason is that there are many ways to draw the same network. However, there will be same network representation drawn from above set of rules which are much easier to follow than the other. In the case of a large network, it is essential that certain 'good habits' be practised to draw an *'easy to follow'* network.

(1) Try to avoid arrows which cross each other.

(2) Use straight arrows.

(3) Do not attempt to represent duration of activity by its arrow length.

(4) Use arrows from left to right (or *right to left*). Avoid mixing two directions, vertical and standing arrows may be used if necessary.

(5) Use dummies freely in rough draft but final network should not have any redundant dummies.

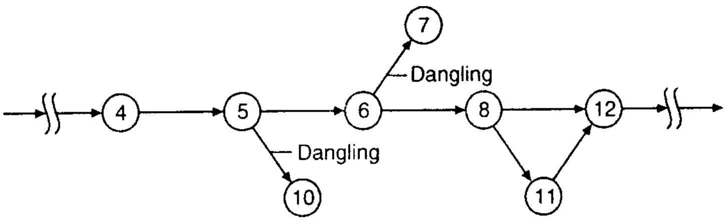
(6) The network has only one entrypoint–called the start event and one point of emergence–called the end event.

In many situations, all these may not be compatible with each activity and some of them are violated. The idea of having a simple network is to facilitate easy reading for all those who are involved in the project.

25.6-1. Common Errors in Drawing Networks

Three types of errors are most commonly observed in drawing network diagrams.

(1) Dangling. To disconnect an activity before the completion of all activities in a network diagram is known as *dangling*. As shown in the figure, activities (5—10) and (6—7) are not the last activities in the network. So the diagram is wrong and indicates the error of dangling.

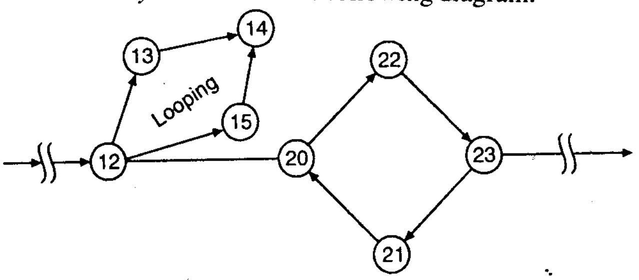


Dangling Error Diagram

Fig. 25.13 (b). Dangling Error Diagram

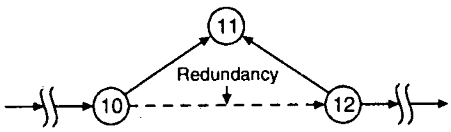
(2) Looping (or Cycling). Looping error is also known as cycling error in a network diagram. Drawing an endless loop in a network is known as an error of looping as shown in the following figure.

(3) Redundancy. Unnecessarily inserting the *dummy activity* in a network logic is known as the error of *redundancy* as shown in the following diagram.



Looping (or Cycling) Error Diagram

Fig. 25.14. Looping (or Cycling) Error Diagram



Redundancy Error

Fig. 25.15. Redundancy Error

25.7. LABELLING : *FULKERSONS'S '1-J' RULE*

For the network representations, it is necessary that various nodes be properly labelled. For convenience, labelling is done on the network diagram. A standard procedure, called the *'I-J' rule* developed by *D.R. Fulkerson*, is most commonly used for this purpose. Main steps of this procedure are :

(*a*) A start event is the one which has arrows emerging from it but none entering it. Find the start event and number it as unity (1).

(*b*) Delete all arrows emerging from all numbered events. This will create at least one new start event out of preceding events.

(*c*) Number all new start events '2', '3', and so on (no definite rule is necessary, but numbering from *'top to bottom'* may facilitate other users in reading the network when there are more than one new start events.

(*d*) Go on repeating steps number (*b*) and (*c*) until the end is reached.

Now consider the network diagram of *Fig*. 25.6. for 'labelling' by *Fulkerson's rule.*

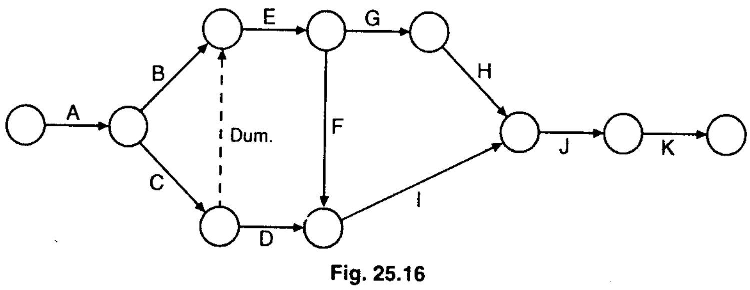


Fig. 25.16

To number the nodes using *Fulkerson's rule*, numbering of nodes 1 and 2 is obvious. Apply step 2. The bottom node is the only node from jobs which is emerging out but none entering it. This is number 3.

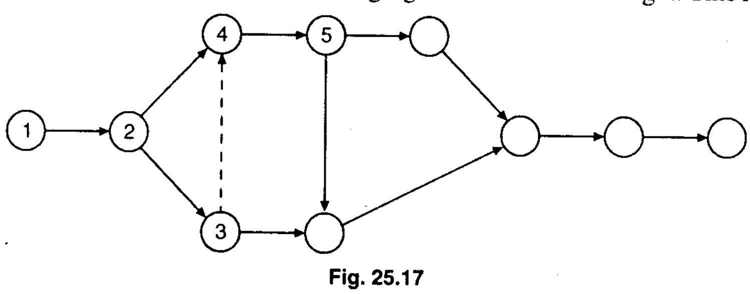


Fig. 25.17

Applying step 2 again, node number 4 is easily obtained and likewise 5 can also be determined.

Using step 2 again, there are two starting points, and either one of them could be numbered 6. Keeping in view the case of numbers 4, 5 and 6 all in a row, the top node is numbered 6. Rest of the numbering procedure is simple and the complete network with numbers is shown in Fig 25.18.

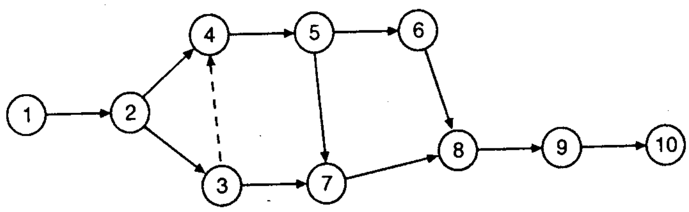


Fig. 25.18

The importance of good numbering procedure can hardly be claimed in a simple network, but the Fulkerson's procedure has certain distinct advantages when the network is large. First, the Fulkerson's procedure will always detect a close loop in the network if there is any. In network methods, a close loop represents an impossible event. Second, numbers are smaller toward the start side and become larger on the end. A third advantage will become apparent when a matrix representation of the network is brought for computerization.

Q. Identity the rules for construction of a network.

A project consists of 12 jobs levelled A to L. The order in which the jobs are to be carried out are : Job A comes first and procedes B, C and D. Both B and C must be done before E starts, and C and D must precede F, G and H can start as soon as D is completed. Job J succeds E, F and G. Jobs I and K can start after H and I are completed. Job L comes after J and K.

(i) Draw the arrow diagram for the project using as few dummies as possible.

(ii) Number the nodes using Fulkerson's rule. **[Bhubnashwar (IT) 2004; VTU (BE Mech.) 2003]**

25.8. TIME ESTIMATES AND CRITICAL PATH IN NETWORK ANALYSIS

Once the network of a project is constructed, the time analysis of the network becomes essential for planning various activities of the project. An activity-time is a forecast of the time an activity is expected to take from its starting point to its completion (under normal conditions).

The main objective of the time analysis is to prepare a planning schedule of the project. The planning schedule should include the following factors :

(1) Total completion time for the project.
(2) Earliest time when each activity can start.
(3) Latest time when each activity can be started without delaying the total project.
(4) Float for each activity, *i.e.. the amount of time by which the completion of an activity can be delayed without delaying the total project completion.*
(5) Identification of critical activities and critical path.

25.8-1. Basic Scheduling Computations

We shall use the following notations for basic scheduling computations.

(i,j) = **Activity (i,j) with tail event i and head event j**
T_E or E_i = **Earliest occurrence time of event i**
T_L or L_j = **Latest allowable occurrence time of event j**
D_{ij} = **Estimated completion time of activity (i,j)**
$(Es)_{ij}$ = **Earliest starting time of activity (i,j)**
$(Ef)_{ij}$ = **Earliest finishing time of activity (i,j)**
$(Ls)_{ij}$ = **Latest starting time of activity (i,j)**
$(Lf)_{ij}$ = **Latest finishing time of activity (i,j).**

The basic scheduling computations can be put under the following three groups :

25.8-2. Forward Pass Computations (For Earliest Event Time) [Meerut (OR) 2003]

Before starting computations, the occurrence time of *initial* network *event* is fixed. Then, the forward pass computation yields the *earliest start* and *earliest finish* time for each activity (i, j), and indirectly the earliest expected occurrence time for each event. This is mainly done in three steps.

Step 1. The computations begin from the '*start*' node and move towards the '*end*' node. For easiness, the forward pass computations start by assuming the earliest occurrence time of *zero* for the initial project event.

Step 2. (*i*) *Earliest starting time* of activity (i, j) is the earliest event time of the tail end event *i.e.*, $(Es)_{ij} = E_i$.

(*ii*) *Earliest finish time* of activity (i, j) is the earliest starting time + the activity time *i.e.*,

$$(Ef)_{ij} = (Es)_{ij} + D_{ij} \quad \text{or} \quad (Ef)_{ij} = E_i + D_{ij}$$

(*iii*) *Earliest event time* for event j is the *maximum* of the earliest finish times of all activities ending into that event. That is,

$$E_j = \max_i \, [(Ef)_{ij} \text{ for all immediate predecessor of}(i, j)] \quad \text{or} \quad E_j = \max_i \, [E_i + D_{ij}]$$

The computed 'E' values are put over the respective circles representing each event.

25.8-3. Backward Pass Computations (For Latest Allowable Time) [Meerut (OR) 2003]

The latest event times (L) indicates the time by which all activities entering into that event must be completed without delaying the completion of the project. These can be computed by reversing the method of calculation used for earliest event times. This is done in the following steps :

Step 1. For ending event assume $E = L$. Remember that all E's have been computed by *forward pass computations.*

Step 2. *Latest finish time* for activity (i, j) is equal to the *latest event time* of event j, i.e., $(Lf)_{ij} = L_j$.

Step 3. *Latest starting time* of activity (i, j) = the *latest completion time* of (i, j) – *the activity time.*

or $\quad (Ls)_{ij} = (Lf)_{ij} - D_{ij} \quad$ or $\quad (Ls)_{ij} = L_j - D_{ij}$.

Step 4. *Latest event time for event i is* the *minimum* of the *latest start time of* all activities originating from that event, *i.e.*,

$$L_i = \min_j \, [(Ls)_{ij} \text{ for all immediate successors of } (i, j)] = \min_j \, [(Lf)_{ij} - D_{ij}] = \min_j \, [L_j - D_{ij}]$$

The computed 'L' values are put over the respective circles representing each event.

25.8-4. Determination of Floats and Slack Times

When the network diagram is completely drawn, properly labelled, and earliest (E) and latest (L) event times are computed as discussed so far, the next object is to determine the *floats* and *slack times* defined as follows :

There are mainly *three* kinds of floats as given below :

(1) Total float. *The amount of time by which the completion of an activity could be delayed beyond the earliest expected completion time without affecting the overall project duration time.* [VTU (BE Mech.) 2003]

Mathematically, the total float of an activity $(i - j)$ is the difference between the *latest start time* and *earliest start time* of that activity. Hence the *total float* for an activity $(i - j)$, denoted by $(Tf)_{ij}$, can be calculated by the formula :

$$(Tf)_{ij} = (\textit{Latest start} - \textit{Earliest start}) \text{ for activity}(i - j)$$

or $\quad (Tf)_{ij} = (Ls)_{ij} - (Es)_{ij} \quad$ or $\quad (Tf)_{ij} = (L_j - D_{ij}) - E_i$

where E_j, L_j and D_{ij} are defined in **sec.25.8-1.** This is the most important type of float because of concerning with the overall project duration.

(2) Free float. *The time by which the completion of an activity can be delayed beyond the earliest finish time without affecting the earliest start of a subsequent(succeeding) acitivity.* [VTU (BE Mech.) 2003]

Mathematically, the free float for activity (i, j), denoted by $(Ff)_{ij}$, can be calculated by the formula :

$$(Ff)_{ij} = (E_j - E_i) - D_{ij}$$

In other words, *Free float for* $(i - j)$ = *(Earliest time for event j – Earliest time for event i) – Activity time for (i, j)*.

This float is concerned with the commencement of subsequent activity.

Remarks :

1. We know that : $(Tf)_{ij} = (L_j - E_i) - D_{ij}$. But, $L_j \geq E_j$ as latest event time is always greater than or equal to the earliest event time. Therefore,
   $$(Tf)_{ij} \geq (E_j - E_i) - D_{ij} \quad \text{or} \quad (Tf)_{ij} \geq (Ff)_{ij}$$
   Hence for all activities, free float can take values from *zero* up to total *float*, but it cannot *exceed* total float.
2. Free float is very useful for rescheduling the activities with minimum disruption of earlier plans.

(3) **Independent float.** *The amount of time by which the start of an activity can be delayed without effecting the earliest start time of any immediately following activities, assuming that the preceding activity has finished at its latest finish time.* **[VTU (BE Mech.) 2003]**

Mathematically, independent float of an activity (*i, j*) denoted by $(If)_{ij}$ can be calculated by the formula :

$$(If)_{ij} = (E_j - L_i) - D_{ij}$$

The *negative* independent float is always taken *zero*. This float is concerned with prior and subsequent activities.

Remarks :

1. It can be observed that Independent float ≤ Free float ≤ Total float.
2. The concept of floats is useful for the management in representing under utilized resources and flexibility of the schedule and the extent to which the resources will be utilized on different acitivities.
3. The float can be used for re-deployment of resources to level the same or to reduce project duration. However, one should bear in mind that whenever the float in a particular activity is utilized, the float of not only that activity but that of other activities would also change.

(4) Interfering Float **[C.A. (May) 92]**

Utilization of float of an activity may affect the float times of the other activity in the network. Interfering float is that part of total float which causes a reduction in the float of successor activities. It is the difference between the latest finish time of activity in question and the earliest starting time of the following activity or zero whichever is larger. It represents the portion of the float of an activity which cannot be consumed without adversely affecting the float of the subsequent activities.

(5) **Event slacks.** *For any given event, the* ***event slack*** *is defined as the difference between the latest event and earliest event times. Mathematically*, for a given activity (*i, j*),

Head event slack $= L_j - E_j$, ***Tail event slack*** $= L_i - E_i$

All the floats defined earlier can be represented in terms of head and tail event slacks also.

Total float $= L_j - E_i - D_{ij}$

Free float $= E_j - E_i - D_{ij} = (L_j - E_i - D_{ij}) - (L_j - E_j) =$ ***Total float – Head event stack***

Independent float $= E_j - L_i - D_{ij} = (E_j - E_i - D_{ij}) - (L_i - E_i) =$ ***Free float – Tail event slack***

(6) Time scale representation of floats and slacks

The various floats and slacks for an activity $(i - j)$ can be represented on a time scale as shown below :

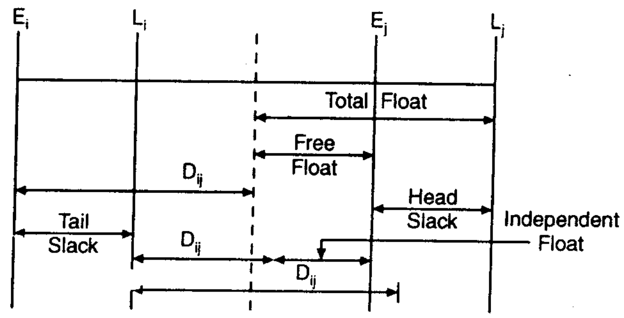
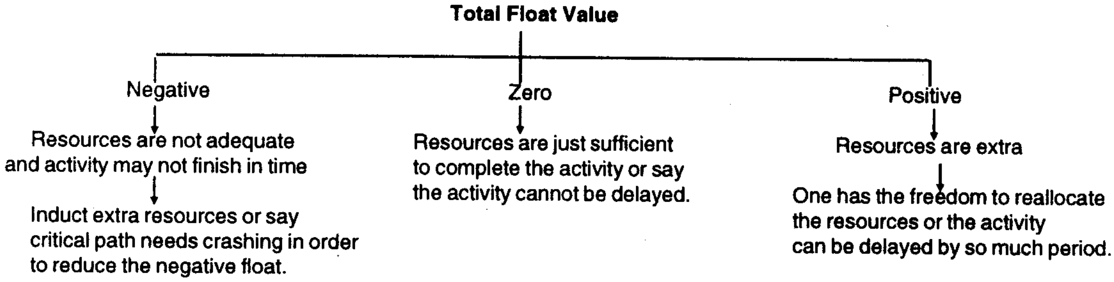


Fig. 25.19.

Conclusions drawn from total float values :

The value of total float for any activity is useful for drawing the following conclusions :



Q. 1. Briefly explain the four types of floats used in Network Analysis. **[CA. (May) 92]**

2. Explain in brief the following terms commonly used in network of PERT/CPM.
(i) Activity (ii) Event (iii) Dummy activity (iv) Path (v) Critical path. **[Meerut (OR) 2003; VTU (BE Mech.) 2002]**

25.8-5. Determination of Critical Path

Before defining *critical path*, let us first discuss about the meaning of *critical event* and *critical activity*.

Critical event. *Since the slack of an event is the difference between the latest and earliest event times, i.e., slack* $(i) = L_i - E_i$, *the events with zero slack times are called critical events.*

In other words, *the event (i) is said to be critical if* $E_i = L_i$.

Critical activity. *Since the difference between the latest start time and earliest start time of an activity is usually called as* ***total float***, *the activities with zero total float are known as* ***critical activities.***

In other words, *an activity is said to be* ***critical*** *if a delay in its start will cause a further delay in the completion date of the entire project.*

Obviously, a non-critical activity is such that the time between its earliest start and its latest completion dates (as allowed by the project) is longer than its actual duration. In this case, non-critical activity is said to have a *slack* or *float* time.

Critical Path. *The sequence of critical activities in a network is called the* ***critical path.*** **[Bhubneshwar (IT) 2004]**

The critical path is the *longest path* in the network from the starting event to ending event and defines the minimum time required to complete the project.

By the term '*path*' we mean a sequence of activities such that it begins at the starting event and end at the final event. The length of a path is the sum of the individual times of the activities lying on the path.

If the activities on critical path are delayed by a day, the project would also be delayed by a day unless the times of the future critical activities are reduced by a day by different means. The critical path is denoted by *double* or *darker* lines to make distinction from the other non-critical paths.

Main features of critical path. The critical path has two main features :

(*i*) If the project has to be shortened, then some of the activities on that path must also be shortened. The application of additional resources on other activities will not give the desired result unless that critical path is shortened first.

(*ii*) The variation in actual performance from the expected activity duration time will be completely reflected in one-to-one fashion in the anticipated completion of the whole project.

The *critical path* identifies all critical activities of the project. The method of determining such a path is explained by the following numerical example.

Example 1. *Consider the following network where nodes have been numbered according to the Fulkerson's rule. Numbers along various activities represent the normal time* (D_{ij}) *required to finish that activity, e.g. activity* (3)—(6) *will take 5 days (months, weeks, hours depending on the time units). For this project, we are*

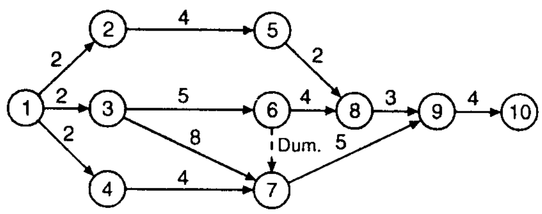


Fig. 25.20.